Liet. matem. rink., **47**, *No.* 4, 2007, 1–15

ON DRIFTLESS ONE-DIMENSIONAL SDE'S WITH RESPECT TO STABLE LEVY PROCESSES

V.P. Kurenok

Department of Natural and Applied Sciences, University of Wisconsin-Green Bay
2420 Nicolet Drive, Green Bay, WI 54311-7001, USA
(e-mail: kurenokv@uwgb.edu)

Abstract. The time-dependent SDE $\mathrm{d}X_t = b(t, X_{t-})\,\mathrm{d}Z_t$ with $X_0 = x_0 \in \mathbb{R}$, and a symmetric α-stable process Z, $1 < \alpha \leqslant 2$, is considered. We study the existence of nonexploding solutions of the given equation through the existence of solutions of the equation $\mathrm{d}A_t = |b|^\alpha(t, \bar{Z} \circ A_t)\,\mathrm{d}t$ in class of time-change processes, where $\bar{Z}$ is a symmetric stable process of the same index α as Z. The approach is based on using the time-change method, Krylov's estimates for stable integrals, and properties of monotone convergence. The main existence result extends the results of Pragarauskas and Zanzotto (2000) for $1 < \alpha < 2$ and those of T. Senf (1993) for $\alpha = 2$.

Keywords: one-dimensional stochastic equations, measurable coefficients, symmetric stable processes, time-change equation, monotone convergence.

Received 06 07 2007

1. INTRODUCTION

We study the one-dimensional stochastic differential equation

$$\mathrm{d}X_t = b(t, X_{t-})\,\mathrm{d}Z_t, \quad t \geqslant 0, \quad X_0 = x_0 \in \mathbb{R}, \tag{1.1}$$

where b: $\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a Borel measurable function, $x_0 \in \mathbb{R}$ is an arbitrary initial value, and Z is a symmetric α-stable process starting at zero, $\alpha \in (1, 2]$. We are interested in conditions ensuring the existence of weak solutions.

The time-independent Eq. (1.1) with $\alpha = 2$ (Z is a Brownian motion) was studied in detail by Engelbert and Schmidt [2]. In particular, they found sufficient and necessary conditions for the existence of solutions. This result was extended later by Zanzotto [12] to the case of a time-independent Eq. (1.1) for symmetric stable processes with $\alpha \in (1, 2]$.

In the case of a time-dependent coefficient b, there are different formulations of sufficient conditions. To state some results for this case, let us introduce the following notation. Define the sets

$$\mathcal{M}_\alpha =: \left\{ y\colon \int_{U(y)} |b|^{-\alpha}(x) l(\mathrm{d}x) = \infty \text{ for any open neighborhood } U(y) \text{ of } y \right\}$$

LMR lmr v.2004/01/22 Prn:14/11/2007; 14:11 F:LMR147.tex; (RRR) p. 2

and

$$\mathcal{N} =: \{y\colon b(y) = 0\},$$

where $l(\mathrm{d}x)$ is the Lebesgue measure on $\mathbb{R}_+ \times \mathbb{R}$. First, the result of Engelbert and Schmidt (correspondingly, of P.A. Zanzotto) for the time-independent case is as follows:

For any initial value $x_0 \in \mathbb{R}$, there exists a solution of (1.1) if and only if

(I) $\mathcal{M}_2 \subseteq \mathcal{N}$ ($\mathcal{M}_\alpha \subseteq \mathcal{N}$, correspondingly).

One says that a measurable function $g\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is Lebesgue locally integrable if $\int_G |g|(y) l(\mathrm{d}y) < \infty$ for any compact set $G \subset \mathbb{R}_+ \times \mathbb{R}$. We shall need the following conditions:

(1) $|b|^{-\alpha}$ is locally integrable with respect to measure l;

(2) $|b|^{\alpha}$ is locally integrable with respect to measure l;

(3) $l_1\{y \in \mathbb{R}\colon \sup_{0\leqslant s\leqslant t} |b(s, y)| < \infty\} > 0$ for all $t > 0$, where l_1 is the Lebesgue measure on $\mathbb{R}$.

Senf [10] considered the time-dependent Eq. (1.1) in the case of Brownian motion and proved that conditions (1), (2), and (3) (for $\alpha = 2$) ensure the existence of a solution for any initial value $x_0 \in \mathbb{R}$. Similar results were also obtained by Rozkosz and Słomiński [9]. Pragarauskas and Zanzotto [7] extended this result to the case of any symmetric stable process with $\alpha \in (1, 2]$.

The goal of this note is to improve the results of Pragarauskas and Zanzotto and those of Senf, respectively, replacing condition (1) by condition (I). We notice that it follows from (1) that $l(\mathcal{N}) = 0$ and $\mathcal{M}_\alpha = \emptyset$, so that automatically $\mathcal{M}_\alpha \subseteq \mathcal{N}$. Therefore, condition (I) is weaker than (1). We shall also mention here the results of Raupach [8], who proved the existence of so-called exploding solutions for Eq. (1.1) with $\alpha = 2$, assuming that b satisfies the conditions (I) and (2). Technically, the approach here is similar to that used in [8], though we consider the nonexploding solutions.

The paper is organized as follows. In Section 2, we show the relationship between the stochastic Eq. (1.1) and the corresponding time-change equation in terms of the existence of a solution. The main result is proved in Section 3 and states sufficient conditions when Eq. (1.1) has a solution for all $1 < \alpha \leqslant 2$. In Appendix, we collected some known facts about monotone convergence and a version of Krylov's estimates for stable integrals.

2. STOCHASTIC AND TIME CHANGE EQUATIONS

We begin with some definitions. As usual, we denote by $\mathbb{D}_{[0,\infty)}(\mathbb{R})$ the Skorokhod space, i.e., the set of all real-valued functions on $[0,\infty)$ with right-continuous trajectories and finite left limits.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space carrying a process Z with $Z_0 = 0$ and $\mathbb{G} = (\mathcal{G}_t)$ be a filtration on $(\Omega, \mathcal{F}, P)$. The notation $(Z, \mathbb{G})$ means that Z is adapted to the filtration $\mathbb{G}$. We call $(Z, \mathbb{G})$ a *symmetric stable process* of index $\alpha \in (0, 2]$ if trajectories of Z belong to $\mathbb{D}$ and

$$\mathbf{E}\Big(\exp\big(ia(Z_t - Z_s)\big)\big|\mathcal{G}_s\Big) = \exp\big(-(t-s)|a|^\alpha\big)$$

for all $t > s \geqslant 0$ and $a \in \mathbb{R}$.

A stochastic process $(X, \mathbb{F})$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ and with trajectories in $\mathbb{D}$ is called a *weak solution* of (1.1) with initial value $x_0 \in \mathbb{R}$ if there exists a symmetric stable process $Z = (Z_t)_{t\geqslant 0}$ with respect to the filtration $\mathbb{F}$ such that $Z_0 = 0$ and

$$X_t = x_0 + \int_0^t b(s, X_{s-})\,\mathrm{d}Z_s, \quad t \geqslant 0, \quad \mathbf{P}\text{-a.s.} \tag{2.1}$$

We are going to show that in order to prove the existence of a solution X of Eq. (2.1) it suffices to construct a time-change process A defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ with filtration $\tilde{\mathbb{F}}$ such that it satisfies the Lebesgue integral equation

$$A_t = \int_0^t |b|^\alpha(s, x_0 + \tilde{Z} \circ A_s)\,\mathrm{d}s, \quad t \geqslant 0, \quad \tilde{\mathbf{P}}\text{-a.s.}, \tag{2.2}$$

where $(\tilde{Z}, \tilde{\mathbb{F}})$ is a symmetric stable process of the same index α, and $\tilde{Z} \circ A$ stands for the superposition of processes $\tilde{Z}$ and A. We call Eq. (2.2) *the time-change equation* associated with (2.1).

Recall that a process A is called an $\tilde{\mathbb{F}}$-time change if it is an increasing right-continuous process with $A_0 = 0$ such that A_t is an $\tilde{\mathbb{F}}$-stopping time for any $t \geqslant 0$ (for details, see [4], Chapter 6). Define $T_t =: \inf\{s \geqslant 0\colon\ A_s > t\}$ called the right-continuous inverse process to A. By definition, T is an increasing process starting at zero. It is easy to see that T is an $\tilde{\mathbb{F}}$-adapted process if and only if A is an $\tilde{\mathbb{F}}$-time change. Instead of Eq. (2.2) for A, we can consider the following equation for the inverse process T:

$$T_t = \int_0^t |b|^{-\alpha}(T_s, x_0 + \tilde{Z}_s)\,\mathrm{d}s, \quad t \geqslant 0, \quad \tilde{\mathbf{P}}\text{-a.s.}, \tag{2.3}$$

where $|b|^{-\alpha} := 1/|b|^\alpha$. We call (2.3) *the inverse time change equation* for (2.1).

In general, Eqs. (2.2) and (2.3) are not equivalent in the sense of existence of solutions. To have the equivalency, we should impose some conditions on the coefficient b (see Lemma 3.2 below). To our knowledge, in all papers dealing with the weak existence of solutions of Eq. (1.1) for $\alpha \in (0, 2]$, Eq. (2.3) was used as a "bridge" for constructing solutions of stochastic equation (cf. [1], [7], [12]). Here we use Eq. (2.2) instead of (2.3) as a basic equation to construct a solution X of (1.1). The following statement justifies this approach.

PROPOSITION 2.1. *Let* $x_0 \in \mathbb{R}$ *and* $\alpha \in (0, 2]$. *If there exists a time-change process* (A_t) *such that* $A_t \in [0, \infty)$ *for all* $t \geqslant 0$ *and it satisfies Eq.* (2.2). *Then there exists a solution* X *of Eq.* (1.1).

Proof. Let A be a solution of Eq. (2.2) with a symmetric α-stable process $(\tilde{Z}, \tilde{\mathbb{F}})$ defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$. For $0 < \alpha < 2$, the process $\tilde{Z}$ is a purely discontinuous semimartingale, so that we can define its characteristic called the jump-measure as $p_{\tilde{Z}}(t, U) := \sum_{s \leqslant t} \mathbf{1}_U(\Delta \tilde{Z}_s)$ for any Borel set U in $\mathbb{R} \setminus \{0\}$, where $\Delta \tilde{Z}_s = \tilde{Z}_s - \tilde{Z}_{s-}$. As is known, this measure is determined by its compensator (dual predictable projection of $p_{\tilde{Z}}$) $q_{\tilde{Z}}(t, U)$ which has the form $q_{\tilde{Z}}(\mathrm{d}t,\ \mathrm{d}x) = \mathrm{d}t \times \frac{\mathrm{d}x}{|x|^{\alpha+1}}$ (see, e.g., [4], Proposition 13.9). For $\alpha = 2$, the process $\tilde{Z}$ is a Brownian motion which can be described as a continuous local martingale with quadratic variation process $\langle \tilde{Z} \rangle_t = t$, $t \geqslant 0$.

Using the time change $(A, \tilde{\mathbb{F}})$, we define the new process $(X, \mathbb{F})$ by

$$X_t := x_0 + \tilde{Z}_{A_t}, \quad \mathcal{F}_t := \tilde{\mathcal{F}}_{A_t}, \ \ t \geqslant 0. \tag{2.4}$$

Now we are going to use time change properties in semimartingales following [3]. Obviously, the process X is a purely discontinuous semimartingale, and its compensator $q_X(\mathrm{d}t,\ \mathrm{d}x)$ can be calculated from $q_{\tilde{Z}}(\mathrm{d}t,\ \mathrm{d}x)$ using the time change $t \to A_t$ (cf. [12], Theorem 2.5). Relation (2.2) yields

$$q_X(\mathrm{d}t,\ \mathrm{d}x) = \left|b(t, X_t)\right|^\alpha \mathrm{d}t \times \frac{\mathrm{d}x}{|x|^{\alpha+1}}.$$

Thus, we have constructed a purely discontinuous semimartingale X with compensator p_X, which is absolutely continuous with respect to the compensator of a symmetric stable process with index α. According to Theorem 2 in [11], there exists (in general, an extended) probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a symmetric stable process Z on it such that

$$X_t = x_0 + \int_0^t b(s, X_{s-})\, \mathrm{d}Z_s. \tag{2.5}$$

For $\alpha = 2$, we use (2.4) to conclude that X is a continuous local martingale with quadratic variation process $\langle X \rangle_t = A_t$, $t \geqslant 0$. Because of (2.2), $\langle X \rangle_t = \int_0^t |b|^\alpha(s, X_s)\, \mathrm{d}s$, so that due to the well-known representation theorem of Doob, there exist an extended probability space and a Brownian motion Z on it such that (2.5) is true as well. Therefore, we have proved that X is a solution of Eq. (1.1) for all $0 < \alpha \leqslant 2$.

3. EXISTENCE OF SOLUTIONS

In this section, we study the existence of solutions of Eq. (1.1) for $1 < \alpha \leqslant 2$. We fix a symmetric stable process $(\tilde{Z}, \tilde{\mathbb{F}})$ defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$, and let $x_0 \in \mathbb{R}$ be an arbitrary initial value.

We proceed as follows: first, we prove the existence of solutions of Eq. (1.1) for lower semicontinuous coefficients b satisfying some additional assumptions and then we extend the result to the coefficients being subject to conditions (I), (2), and (3). Our tools are the time-change method, Krylov's estimates for symmetric stable processes, and some analytic facts about the monotone convergence.

First, we formulate the following version of Krylov's estimates.

LEMMA 3.1. *Let* $g\colon \mathbb{R}_+ \times \mathbb{R} \to [0,\infty]$ *be a measurable function and* A *be a solution of the time-change Eq. (2.2) with symmetric stable process* $(\tilde{Z}, \mathbb{F})$ *and* $x_0 \in \mathbb{R}$ *such that* $A_t < \infty$ *for all* $t \geqslant 0$. *Then, for all* $m \in \mathbb{N}$ *and* $t \geqslant 0$,

$$\begin{aligned}
&\tilde{\mathbf{E}} \int_0^{t\wedge\tau_m(x_0+\tilde{Z}\circ A)} g(s, x_0 + \tilde{Z}\circ A_s)|b|^{\frac{\alpha}{2}}(s, x_0 + \tilde{Z}\circ A_s)\,\mathrm{d}s \\
&\quad \leqslant N\Big(\int_0^t \int_{-m}^m g^2(y) l(\mathrm{d}y)\Big)^{\frac{1}{2}} := N\|g\|_{2,t,m;l},
\end{aligned}$$

where $\tau_m(x_0+\tilde{Z}\circ A) = \inf\{t \geqslant 0\colon\ |x_0 + \tilde{Z}\circ A_t| \geqslant m\}$, *and the constant* N *depends on* α, m, *and* t *only.*

Proof. We define $X_t = x_0 + \tilde{Z}\circ A_t$ and use Proposition 2.1 to conclude that the process X satisfies the equation

$$X_t = x_0 + \int_0^t b(s, X_{s-})\,\mathrm{d}Z_s, \quad t \geqslant 0,$$

with symmetric stable process Z of the same index α. The required estimate follows then from Lemma 4.3.

To prove the existence of solutions of Eq. (2.2) for lower semicontinuous, nondegenerate coefficients b, we shall use the inverse time-change Eq. (2.3) and the following Lemma 3.2. Notice first that a stochastic process Y is said to be *nonexploding* if $Y_t < \infty$ for all $t \geqslant 0$.

LEMMA 3.2. *Assume that* $|b| > r > 0$, *where* r *is a constant, and the condition* (3) *is satisfied. Then, the time-change Eq.* (2.2) *has a nonexploding solution* A *if and only if Eq.* (2.3) *has a nonexploding solution* T.

Proof. Assume first that A is a nonexploding time-change process satisfying (2.2) and T is its inverse. Notice that T is defined for all $t \geqslant 0$, because A is strictly increasing and $A_\infty := \lim_{t\to\infty} A_t = \infty$ due to the assumptions. Moreover, $A_{T_t} = t$, $t \geqslant 0$.

Because of $|b| > r > 0$ and by properties of change of variables in the Lebesgue–Stieltjes integral, we have

$$T_t = \int_0^{T_t} \frac{|b(s, x_0 + \tilde{Z}\circ A_s)|^\alpha}{|b(s, x_0 + \tilde{Z}\circ A_s)|^\alpha}\,\mathrm{d}s = \int_0^{T_t} \big|b(s, x_0 + \tilde{Z}\circ A_s)\big|^{-\alpha}\,\mathrm{d}A_s$$

$$= \int_0^t \big|b(T_s, x_0 + \tilde{Z}_s)\big|^{-\alpha}\,\mathrm{d}s$$

for all $t \geqslant 0$. Therefore, T is a solution of Eq. (2.3).

Now let T be a nonexploding solution of Eq. (2.3). First, we notice that T is a strictly increasing process because $|b|^{-\alpha} > 0$. Thus A is a well-defined continuous process on the interval $[0, T_\infty)$. In fact, $T_\infty = \infty$ implying that A is nonexploding and $T_{A_t} = t$ for all $t \geqslant 0$. Indeed, to see that $T_\infty = \infty$, let us assume that $\tilde{\mathbf{P}}(T_\infty < \infty) > 0$, that is, there exists $t^\star > 0$ such that $\tilde{\mathbf{P}}(T_\infty < t^\star) > 0$. Set

$$h(x) = \inf_{s \in [0, t^\star]} |b|^{-\alpha}(s, x).$$

The condition (3) yields that

$$l_1\{x\colon\ h(x) > 0\} > 0,$$

where l_1 is the Lebesgue measure on $\mathbb{R}$. Using Corollary 2.3 from [1] (see also [12]), we conclude that

$$\tilde{\mathbf{P}}\Big(\int_0^\infty h(x_0 + \tilde{Z}_s)\,\mathrm{d}s = \infty\Big) = 1.$$

On the other hand, on the set $\{T_\infty < t^\star\}$, we have

$$\infty = \int_0^\infty h(x_0 + \tilde{Z}_s)\,\mathrm{d}s \leqslant \int_0^\infty |b|^{-\alpha}(T_s, x_0 + \tilde{Z}_s)\,\mathrm{d}s \leqslant t^\star,$$

which is a contradiction to the assumption. Hence, $T_\infty = \infty$ $\tilde{\mathbf{P}}$-a.s.

Finally, for all $t \geqslant 0$, then we have

$$A_t = \int_0^{A_t} \frac{|b(T_s, x_0 + \tilde{Z}_s)|^\alpha}{|b(T_s, x_0 + \tilde{Z}_s)|^\alpha}\,\mathrm{d}s = \int_0^{A_t} \big|b(T_s, x_0 + \tilde{Z}_s)\big|^\alpha\,\mathrm{d}T_s$$
$$= \int_0^t \big|b(s, x_0 + \tilde{Z} \circ A_s)\big|^\alpha\,\mathrm{d}s,$$

proving that A is a solution of Eq. (2.2).

PROPOSITION 3.3. *Let b be a lower semicontinuous function satisfying conditions* (2) *and* (3) *such that $|b|^\alpha > r > 0$, where r is a constant. Then, Eq.* (2.2) *has a solution.*

Proof. According to Lemma 4.2, there is a sequence of functions b_n, $n = 1, 2, \ldots,$ such that

$$\big|b_n(y_1) - b_n(y_2)\big| \leqslant K_n(r)|y_1 - y_2| \tag{3.1}$$

for all $y_1, y_2 \in \mathbb{R}_+ \times \mathbb{R}$, where $K_n(r) > 0$ is a constant. Moreover, b_n converge pointwise to $|b|^\alpha$ and $0 < r_1 \leqslant b_1 < b_2 < \ldots \leqslant |b|^\alpha$, where r_1 is a constant depending on r and α only. For every but fixed n, the coefficient b_n is bounded and also nondegenerate. Let $b_n^{-1} := 1/b_n$, $n = 1, 2 \ldots$. From (3.1) it follows that, for all $y_1, y_2 \in \mathbb{R}_+ \times \mathbb{R}$, the sequence of functions b_n^{-1} satisfies the condition

$$\left|b_n^{-1}(y_1) - b_n^{-1}(y_2)\right| \leqslant K_n(r) r_1^{-2} |y_1 - y_2|.$$

Therefore, for any $n \in \mathbb{N}$, b_n^{-1} is a globally Lipschitz continuous, bounded, and nondegenerate function. Clearly, the sequence b_n^{-1} converges monotone pointwise to $|b|^{-\alpha}$. According to Theorem 3.1 in [1], for any $n = 1, 2, \ldots$, there exists a nonexploding solution T^n of the inverse time-change equation

$$T_t^n = \int_0^t b_n^{-1}(T_s^n, x_0 + \tilde{Z}_s)\, \mathrm{d}s. \tag{3.2}$$

From the properties of the sequence (b_n) it follows that all functions b_n satisfy assumption (3). Additionally, using Eq. (3.2) instead of Eq. (2.3), as in the proof of Lemma 3.2, one can show that $T_\infty^n = \infty$ for all $n \in \mathbb{N}$. By Lemma 3.2 again one has then that, for any $n \in \mathbb{N}$, there exists a nonexploding time-change process A^n, the right-inverse to T^n, such that

$$A_t^n = \int_0^t b_n(s, x_0 + \tilde{Z} \circ A_s^n)\, \mathrm{d}s, \quad t \geqslant 0. \tag{3.3}$$

Moreover, due to Lemma 4.1, $A_t^n \leqslant A_t^{n+1}$ for all $t \geqslant 0$ and $n = 1, 2, \ldots$ $\tilde{\mathbf{P}}$-a.s. Define $A_t = \ sup_{n\in\mathbb{N}} A_t^n$. Obviously, the process A is an $\tilde{\mathbb{F}}$-time change such that $A_\infty = \lim_{t\to\infty} A_t = \infty$. Next, we are going to show that the process T, which is the inverse to A, is a solution of Eq. (2.3) with coefficient $|b|^{-\alpha}$. This would mean again by Lemma 3.2 that A solves the time-change Eq. (2.2).

From the definition of T^n it follows that T is continuous, because A is strictly increasing and $T_t = \lim_{n\in\mathbb{N}} T_t^n$. Note that $\tau_m(x_0 + \tilde{Z}) \uparrow \infty$ as $m \to \infty$ $\tilde{\mathbf{P}}$-a.s. Therefore, to show that T satisfies (2.3) it suffices to verify that, for all $\varepsilon > 0$, $m \in \mathbb{N}$, and $t \geqslant 0$,

$$\tilde{\mathbf{P}}\left(\left|T_{t\wedge\tau_m(x_0+\tilde{Z})} - \int_0^{t\wedge\tau_m(x_0+\tilde{Z})} |b|^{-\alpha}(T_s, x_0 + \tilde{Z}_s)\, \mathrm{d}s\right| > \varepsilon\right) = 0.$$

By the Chebyshev inequality,

$$\begin{aligned}
&\tilde{\mathbf{P}}\left(\left|T_{t\wedge\tau_m(x_0+\tilde{Z})} - \int_0^{t\wedge\tau_m(x_0+\tilde{Z})} |b|^{-\alpha}(T_s, x_0 + \tilde{Z}_s)\, \mathrm{d}s\right| > \varepsilon\right) \\
&\quad \leqslant \frac{1}{\varepsilon} \tilde{\mathbf{E}} \left|T_{t\wedge\tau_m(x_0+\tilde{Z})} - T^n_{t\wedge\tau_m(x_0+\tilde{Z})}\right|
\end{aligned}$$

LMR lmr v.2004/01/22 Prn:14/11/2007; 14:11 F:LMR147.tex; (RRR) p. 8

$$
\begin{aligned}
&+\frac{1}{\varepsilon}\tilde{\mathbf{E}}\int_0^{t\wedge\tau_m(x_0+\tilde{Z})}\left|b_k^{-1}-|b|^{-\alpha}\right|(T_s,x_0+\tilde{Z}_s)\,\mathrm{d}s\\
&+\frac{1}{\varepsilon}\tilde{\mathbf{E}}\int_0^{t\wedge\tau_m(x_0+\tilde{Z})}\left|b_k^{-1}(T_s^n,x_0+\tilde{Z}_s)-b_k^{-1}(T_s,x_0+\tilde{Z}_s)\right|\mathrm{d}s\\
&+\frac{1}{\varepsilon}\tilde{\mathbf{E}}\int_0^{t\wedge\tau_m(x_0+\tilde{Z})}\left|b_n^{-1}-|b|^{-\alpha}\right|(T_s^n,x_0+\tilde{Z}_s)\,\mathrm{d}s\\
&+\frac{1}{\varepsilon}\tilde{\mathbf{E}}\int_0^{t\wedge\tau_m(x_0+\tilde{Z})}\left|b_k^{-1}-|b|^{-\alpha}\right|(T_s^n,x_0+\tilde{Z}_s)\,\mathrm{d}s\\
&=\Delta_1+\Delta_2+\Delta_3+\Delta_4+\Delta_5,
\end{aligned}
$$

where $k\in\mathbb{N}$. Then $\Delta_1\to 0$ as $n\to\infty$, since $\lim_{n\to\infty}T^n_{t\wedge\tau_m(x_0+\tilde{Z})}=T_{t\wedge\tau_m(x_0+\tilde{Z})}$ $\tilde{\mathbf{P}}$-a.s. and T^n, $n=1,2\ldots$, is a bounded sequence. Δ_2 converges to zero as $k\to\infty$ by the Lebesgue theorem for uniformly bounded integrals. $\lim_{n\to\infty}\Delta_3=0$ because of the Lebesgue majorized convergence theorem. To prove that $\Delta_4\to 0$ as $n\to\infty$ and $\Delta_5\to 0$ as $k\to\infty$, we need the following variant of Krylov's estimates.

LEMMA 3.4. *Let h: $\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}_+$ be a Borel measurable function. Then, for any $t\geqslant 0$ and $n,m\in\mathbb{N}$, there exists a constant $N=N(\alpha,m,t)$ such that*

$$
\tilde{\mathbf{E}}\int_0^{\tau_m(x_0+\tilde{Z})\wedge A_t^n}h(T_s^n,x_0+\tilde{Z}_s)\,\mathrm{d}s\leqslant N\|h|b|^{\frac{\alpha}{2}}\|_{2,t,m;l}. \tag{3.4}
$$

Proof. Notice that $A^n_{t\wedge\tau_m(x_0+\tilde{Z}\circ A^n)}=A_t^n\wedge\tau_m(x_0+\tilde{Z})$. Making the time change $s\to A_s^n$ in the integral on the left-hand side of (3.4) and using the estimate in Lemma 3.1, we obtain

$$
\begin{aligned}
\tilde{\mathbf{E}}\int_0^{\tau_m(x_0+\tilde{Z})\wedge A_t^n}h(T_s^n,x_0+\tilde{Z}_s)\,\mathrm{d}s&=\tilde{\mathbf{E}}\int_0^{t\wedge\tau_m(x_0+\tilde{Z}\circ A^n)}h(s,x_0+\tilde{Z}\circ A_s^n)\,\mathrm{d}A_s^n\\
&\leqslant\tilde{\mathbf{E}}\int_0^{t\wedge\tau_m(x_0+\tilde{Z}\circ A^n)}\big(h|b|^{\frac{\alpha}{2}}b_n^{\frac{\alpha}{2}}\big)(s,x_0+\tilde{Z}\circ A_s^n)\,\mathrm{d}s\leqslant N\|h|b|^{\frac{\alpha}{2}}\|_{2,t,m;l}.
\end{aligned}
$$

Since $b_n\geqslant r_1>0$ for all $n=1,2,\ldots$, we conclude that $A_t^n\geqslant r_1t$ and hence $t\leqslant A_{tr_1^{-1}}$. Now applying Lemma 3.4 yields

$$
\Delta_4=\frac{1}{\varepsilon}\tilde{\mathbf{E}}\int_0^{t\wedge\tau_m(x_0+\tilde{Z})}\left|b_n^{-1}-|b|^{-\alpha}\right|(T_s^n,x_0+\tilde{Z}_s)\,\mathrm{d}s
$$

$$\leqslant \frac{1}{\varepsilon}\mathbf{E}\int_0^{\tau_m(x_0+\tilde{Z})\wedge A^n_{tr^{-1}}} \big||b|^{-\alpha}-b_n^{-1}\big|(T_s^n, x_0+\tilde{Z}_s)\,\mathrm{d}s$$

$$\leqslant \frac{1}{\varepsilon}N\big\|\big(|b|^{-\alpha}-b_n^{-1}\big)|b|^{\frac{\alpha}{2}}\big\|_{2,tr^{-1},m;l}$$

and

$$\Delta_5 \leqslant \frac{1}{\varepsilon}N\big\|\big(|b|^{-\alpha}-b_k^{-1}\big)|b|^{\frac{\alpha}{2}}\big\|_{2,tr^{-1},m;l}.$$

The right-hand sides of two last inequalities converge to zero as $n\to\infty$ and $k\to\infty$, respectively, since $|b|^{\alpha}$ is a locally integrable function and the expressions $(|b|^{-\alpha}-b_n^{-1})$ and $(|b|^{-\alpha}-b_k^{-1})$ are bounded for all n and k. Thus, we have shown that T is a solution of Eq. (2.3), and the proof of Proposition is finished.

The following statement is then a direct consequence of Propositions 2.1 and 3.3.

THEOREM 3.5. *Let b be a lower semicontinuous function satisfying conditions* (2) *and* (3) *such that $|b|^{\alpha}>r>0$, where r is a constant. Then, for any $x_0\in\mathbb{R}$, there exists a solution of Eq.* (1.1).

Now we are ready to state the main existence result.

THEOREM 3.6. *Let $b\colon\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ be a measurable function satisfying conditions* (I), (2), *and* (3). *Then, for any initial value $x_0\in\mathbb{R}$, there exists a solution of Eq.* (1.1).

Proof. According to Proposition 2.1, it suffices to show that there exists a time-change process A satisfying (2.2).

We first notice that the condition $\mathcal{M}_\alpha\subseteq\mathcal{N}$ can be replaced by $\mathcal{M}_\alpha=\mathcal{N}$. To see this, let $\tilde{b}:=b+\mathbf{1}_{\mathcal{N}\setminus\mathcal{M}_\alpha}$. Obviously, $\mathcal{N}(\tilde{b})=\mathcal{M}_\alpha(\tilde{b})=\mathcal{M}_\alpha(b)$ and $\tilde{b}=b$ l-a.s., since $l(\mathcal{N}\setminus\mathcal{M}_\alpha)=0$. Suppose that $(X,\mathbb{F})$ is a (nonexploding) solution of Eq. (1.1) with coefficient $\tilde{b}$ and symmetric stable process $(Z,\mathbb{F})$ defined on a probability space $(\Omega,\mathcal{F},\mathbf{P})$ with filtration $\mathbb{F}$. To show that X also is a solution of (1.1) with coefficient b, we apply Lemma 4.3 to the coefficient $(\tilde{b}-b)$ to get for all $t\geqslant 0$, $\varepsilon>0$, and $m\in\mathbb{N}$,

$$\mathbf{P}\Big(\int_0^t|\tilde{b}-b|^{\alpha}(s,X_s)\,\mathrm{d}s>\varepsilon\Big)\leqslant\frac{1}{\varepsilon}\mathbf{E}\int_0^{t\wedge\tau_m(X)}|\tilde{b}-b|^{\alpha}(s,X_s)\,\mathrm{d}s+\frac{1}{\varepsilon}\mathbf{P}(\tau_m(X)>t)$$

$$=\frac{1}{\varepsilon}\mathbf{E}\int_0^{t\wedge\tau_m(X)}\mathbf{1}_{\mathcal{N}\setminus\mathcal{M}_\alpha}(s,X_s)\,\mathrm{d}s+\frac{1}{\varepsilon}\mathbf{P}(\tau_m(X)>t)$$

$$\leqslant\frac{1}{\varepsilon}N\big\||\tilde{b}|^{-\frac{\alpha}{2}}\mathbf{1}_{\mathcal{N}\setminus\mathcal{M}_\alpha}\big\|_{2,t,m;l}+\frac{1}{\varepsilon}\mathbf{P}(\tau_m(X)>t)$$

$$\leqslant\frac{1}{\varepsilon}N\|\mathbf{1}_{\mathcal{N}\setminus\mathcal{M}_\alpha}\|_{2,t,m;l}+\frac{1}{\varepsilon}\mathbf{P}(\tau_m(X)>t)=\frac{1}{\varepsilon}\mathbf{P}(\tau_m(X)>t).$$

LMR lmr v.2004/01/22 Prn:14/11/2007; 14:11 F:LMR147.tex; (RRR) p. 10

Since for fixed t there is $m \in \mathbb{N}$ such that $\mathbf{P}(\tau_m(X) > t)$ can be made arbitrarily small, the stochastic integral $\int_0^t (\tilde{b} - b)(s, X_s)\,\mathrm{d}Z_s$ exists and is equal to zero. It follows then that

$$X_t = x_0 + \int_0^t \tilde{b}(s, X_s)\,\mathrm{d}Z_s = x_0 + \int_0^t b(s, X_s)\,\mathrm{d}Z_s \quad \text{for all } t \geqslant 0.$$

Therefore, we can assume that $\mathcal{M}_\alpha = \mathcal{N}$.

By Lemma 4.4, there exists a sequence $\{\bar{b}_n\}$, $n = 1, 2, \ldots,$ of lower semicontinuous and locally integrable functions such that $\mathcal{N}(\bar{b}_n) = \mathcal{N}(b), \bar{b}_1 \geqslant \bar{b}_2 \geqslant \ldots |b|^\alpha$, and $\|\bar{b}_n - |b|^\alpha\|_{2,t,m;\mu} \to 0$ as $n \to \infty$ for all $t \geqslant 0$ and $m \in \mathbb{N}$, where μ is defined in (4.4). Let $b_n := \bar{b}_n + \frac{1}{n}$, $n = 1, 2, \ldots$. For all n, $b_n \geqslant \frac{1}{n} > 0$. Thus, b_n is nondegenerate and also lower semicontinuous. Using the inequalities above, we obtain

$$\begin{aligned}
\int_0^t \int_{-m}^m (b_n - |b|^\alpha)^2 b_n^{-1}\,\mathrm{d}l &= \int_0^t \int_{-m}^m \Big(\bar{b}_n + \frac{1}{n} - |b|^\alpha\Big)^2 \Big(\bar{b}_n + \frac{1}{n}\Big)^{-1}\,\mathrm{d}l \\
&\leqslant 2\int_0^t \int_{-m}^m (\bar{b}_n - |b|^\alpha)^2 \Big(\bar{b}_n + \frac{1}{n}\Big)^{-1}\,\mathrm{d}l + 2\int_0^t \int_{-m}^m \frac{1}{n^2}\Big(\bar{b}_n + \frac{1}{n}\Big)^{-1}\,\mathrm{d}l \\
&\leqslant 2\int_0^t \int_{-m}^m (\bar{b}_n - |b|^\alpha)^2 |b|^{-\alpha}\,\mathrm{d}l + \frac{4}{n} tm.
\end{aligned}$$

Passing to the limit $n \to \infty$ in the above inequality, we establish that

$$\big\|\big(b_n - |b|^\alpha\big) b_n^{-1/2}\big\|_{2,t,m;l} \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

Let $(\tilde{Z}, \tilde{\mathbb{F}})$ be a symmetric stable process defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ and $x_0 \in \mathbb{R}$. By definition of the sequence $\{b_n\}$, one has $b_1 > b_2 > \ldots \geqslant |b|^\alpha$. Hence, by Proposition 3.3 and Lemma 4.1, there exists a monotone sequence of time-change processes $A_t^n \geqslant A_t^{n+1}$, $t \geqslant 0$, such that A^n is a solution of (2.2) with coefficient b_n and process $(\tilde{Z}, \tilde{\mathbb{F}})$. Moreover, for any $n \in \mathbb{N}$ and $t \geqslant 0$, $A_t^n < \infty$ and $A_\infty^n = \infty$. Define $A_t := \inf_{n \in \mathbb{N}} A_t^n$. It follows then that A is a right-continuous time-change process with respect to $\mathbb{F}^{\tilde{Z}}$ and $A_\infty = \infty$. It remains to show that A satisfies Eq. (2.2). For this it suffices to show that

$$\tilde{\mathbf{P}}\Big(\Big|A_t - \int_0^t |b|^\alpha(s, x_0 + \tilde{Z} \circ A_s)\,\mathrm{d}s\Big| > \varepsilon\Big) = 0$$

for all $\varepsilon > 0$ and $t > 0$. Using the Chebyshev inequality, we obtain

$$\begin{aligned}
&\tilde{\mathbf{P}}\Big(\Big|A_t - \int_0^t |b|^\alpha(s, x_0 + \tilde{Z} \circ A_s)\,\mathrm{d}s\Big| > \varepsilon\Big) \leqslant \tilde{\mathbf{P}}\Big(|A_t - A_t^n| > \frac{\varepsilon}{3}\Big) \\
&\quad + \frac{3}{\varepsilon}\tilde{\mathbf{E}} \int_0^{t \wedge \tau_m(x_0 + \tilde{Z} \circ A^n)} \big|b_n - |b|^\alpha\big|(s, x_0 + \tilde{Z} \circ A_s^n)\,\mathrm{d}s
\end{aligned}$$

$$+\frac{3}{\varepsilon}\tilde{\mathbf{E}}\int_0^t \big||b|^\alpha(s,x_0+\tilde{Z}\circ A_s^n)-|b|^\alpha(s,x_0+\tilde{Z}\circ A_s)\big|\,\mathrm{d}s+\tilde{\mathbf{P}}\big(\tau_m(\tilde{Z}\circ A^n)>t\big)$$
$$=\Delta_5+\Delta_6+\Delta_7+\tilde{\mathbf{P}}\big(\tau_m(\tilde{Z}\circ A^n)>t\big).$$

The convergence $\Delta_5\to 0$ as $n\to\infty$ is clear, since $A_t^n\to A_t$ as $n\to\infty$ $\tilde{\mathbf{P}}$-a.s. By Lemma 3.1 and (3.5) we have

$$\Delta_6\leqslant\frac{3}{\varepsilon}\tilde{\mathbf{E}}\int_0^{t\wedge\tau_m(x_0+\tilde{Z}\circ A^n)}\big|b_n-|b|^\alpha\big|b_n^{1/2}b_n^{-1/2}(s,x_0+\tilde{Z}\circ A_s^n)\,\mathrm{d}s$$
$$\leqslant\frac{3}{\varepsilon}N\big\|\big(b_n-|b|^\alpha\big)b_n^{-1/2}\big\|_{2,t,m;l},$$

which converges to zero for $n\to\infty$. By properties of the processes $\tilde{Z}$ and $A^n, n=1,2,\ldots$, there exits $m\in\mathbb{N}$ such that the term $\tilde{\mathbf{P}}(\tau_m(\tilde{Z}\circ A^n)>t)$ can be made arbitrarily small for all $n=1,2,\ldots$. It remains to show that $\Delta_7\to 0$ as $n\to\infty$.

Consider the functions

$$b_{n,k}(x):=\inf_{y\in\mathbb{R}_+\times\mathbb{R}}\big(\bar{b}_n(y)+k\|x-y\|_{\mathbb{R}_+\times\mathbb{R}}\big),\quad n,k\in\mathbb{N}.$$

These functions satisfy $\mathcal{N}(b_{n,k})=\mathcal{N}(\bar{b}_n)=\mathcal{N}(b)$ and are continuous. By construction, the sequence $b_{n,k}$ converges to $\bar{b}_n$ pointwise and in the $\|\cdot\|_{2,t,m;\mu}$-norm as $k\to\infty$. Moreover, $\bar{b}_n$ converges to $|b|^\alpha$ in the same sense as $n\to\infty$. Therefore, there exists a diagonal sequence $a_p=b_{n(p),k(p)}$ such that the functions a_p, $p=1,2,\ldots$, are continuous, a_p converges to $|b|^\alpha$ as $p\to\infty$ in the corresponding L_2-norm, and $\mathcal{N}(a_p)=\mathcal{N}(b)$. Hence,

$$\Delta_7\leqslant\frac{3}{\varepsilon}\tilde{\mathbf{E}}\int_0^{t\wedge\tau_m(\tilde{Z}\circ A^n)}\big||b|^\alpha-a_p\big|(s,x_0+\tilde{Z}\circ A_s^n)\,\mathrm{d}s$$
$$+\frac{3}{\varepsilon}\tilde{\mathbf{E}}\int_0^{t\wedge\tau_m(\tilde{Z}\circ A)}\big||b|^\alpha-a_p\big|(s,x_0+\tilde{Z}\circ A_s)\,\mathrm{d}s$$
$$+\frac{3}{\varepsilon}\tilde{\mathbf{E}}\int_0^{t}\big|a_p(s,x_0+\tilde{Z}\circ A_s^n)-a_p(s,x_0+\tilde{Z}\circ A_s)\big|\,\mathrm{d}s$$
$$+\tilde{\mathbf{P}}\big(\tau_m(\tilde{Z}\circ A^n)>t\big)+\tilde{\mathbf{P}}\big(\tau_m(\tilde{Z}\circ A)>t\big)$$
$$=\Delta_8+\Delta_9+\Delta_{10}+\tilde{\mathbf{P}}\big(\tau_m(\tilde{Z}\circ A^n)>t\big)+\tilde{\mathbf{P}}\big(\tau_m(\tilde{Z}\circ A)>t\big).$$

Again, as before, the last two terms can be made arbitrarily small for all $n\in\mathbb{N}$ by choosing m large enough. Using Lemma 4.3, we further obtain that

$$\Delta_8\leqslant N\big\|\big(|b|^\alpha-a_p\big)b_n^{-1}\big\|_{2,t,m;l}\leqslant N\big\||b|^\alpha-a_p\big\|_{2,t,m;\mu}\to 0\quad\text{as}\quad p\to\infty.$$

The term Δ_{10} converges to zero as $n \to \infty$ by majorized convergence. It remains to verify that $\Delta_9 \to 0$ as $p \to \infty$. For this, we need to establish the following Krylov's estimate:

$$\tilde{\mathbf{E}} \int_0^{t \wedge \tau_m(\tilde{Z} \circ A)} \big| |b|^\alpha - a_p \big| (s, x_0 + \tilde{Z} \circ A_s) \, \mathrm{d}s \leqslant N \big\| |b|^\alpha - a_p \big\|_{2,t,m;\mu} \tag{3.6}$$

with constant N depending on $t, m,$ and α only. Fix p and let $h := ||b|^\alpha - a_p|$. First we notice that applying Lemma 3.1 yields

$$\tilde{\mathbf{E}} \int_0^{t \wedge \tau_m(\tilde{Z} \circ A^n)} h(s, x_0 + \tilde{Z} \circ A_s^n) \, \mathrm{d}s \leqslant N \|h\|_{2,t,m;\mu}. \tag{3.7}$$

To get the desired estimate (3.6) from estimate (3.7), we shall use the following monotone approximation of h. Clearly, h is a lower semicontinuous and locally integrable function with $\mathcal{N}(h) = \mathcal{M}(h)$. According to Lemma 4.4, there exist functions $h_k, k = 1, 2, \ldots,$ such that $h_k \downarrow h$ pointwise and $\|h_k - h\|_{2,t,m;\mu} \to 0$ as $k \to \infty$. Moreover, by Lemma 4.2, for any k, there exists a sequence of globally Lipschitz continuous functions $h_{k,q}, q = 1, 2, \ldots,$ such that $h_{k,q} \uparrow h_k$ pointwise as $q \to \infty$. Then applying Lemma 3.1 to $h_{k,q}$, we get

$$\tilde{\mathbf{E}} \int_0^{t \wedge \tau_m(\tilde{Z} \circ A^n)} h_{k,q}(s, x_0 + \tilde{Z} \circ A_s^n) \, \mathrm{d}s \leqslant N \|h_{k,q}\|_{2,t,m;\mu}$$

and by passing to the limit $n \to \infty$

$$\tilde{\mathbf{E}} \int_0^{t \wedge \tau_m(\tilde{Z} \circ A)} h_{k,q}(s, x_0 + \tilde{Z} \circ A_s) \, \mathrm{d}s \leqslant N \|h_{k,q}\|_{2,t,m;\mu}. \tag{3.8}$$

It remains to notice that we obtain estimate (3.6) from inequality (3.8) by passing first to the limit as $q \to \infty$ and then as $k \to \infty$ and using the monotone convergence theorems. The proof is complete.

Remark 3.7. Finally, we would like to make a comment on the existence of nontrivial and trivial solutions. One says that a solution X is trivial if $\mathbf{P}(X_t = x_0, t \geqslant 0) = 1$; otherwise, it is said to be nontrivial. The conditions of Theorem 3.6 do not guarantee the existence of a nontrivial solution *for an arbitrary initial value* $x_0 \in \mathbb{R}$. For example, if we take $b(t, x) = 0$, $(t, x) \in [0, \infty) \times \mathbb{R}$, which obviously satisfies all conditions of Theorem 3.6, there will exist only a trivial solution $X_t = x_0$ for any initial value $x_0 \in \mathbb{R}$. On another hand, as is shown in [7], conditions (1), (2), and (3) will be ones sufficient for the existence of a nontrivial solution for any initial value. Condition (1), being stronger than condition (I), implies that $l(N) = 0$. However, the later condition is not necessary for the existence of a nontrivial solution for any initial value as the following example shows. Let

$$b(t,x)=\begin{cases}1+t^2+x^2, & (t,x)\in[0,1)\times\mathbb{R},\\ 0, & \text{otherwise.}\end{cases}$$

Then $l(N)>0$, and the function b satisfies assumptions (I), (2), and (3) of Theorem 3.6 but does not satisfy assumption (1). However, it is not difficult to see that any solution of the stochastic equation will be nontrivial on the interval $[0,1)$ and hence nontrivial on $[0,\infty)$.

4. APPENDIX

We collected here some analytical facts about the monotone approximation of functions and a version of Krylov's estimates for stable integrals proven in other sources.

Let $f\colon[0,\infty)\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Consider the Cauchy problem for the first-order differential equation

$$\frac{\mathrm{d}y(s)}{\mathrm{d}s}=f(s,y(s)),\quad y(0)=x_0,\ \ x_0\in\mathbb{R}. \tag{4.1}$$

The following fact is well known in the theory of ordinary differential equations (see also Proposition 22 in [8]).

LEMMA 4.1. *Suppose that f_1 and f_2 are continuous functions with $f_1<f_2$ and that y_1 and y_2 are two corresponding solutions of problem* (4.1). *Then $y_1(s)<y_2(s)$ for all $s<\delta_1\wedge\delta_2$, where δ_1 and δ_2 are explosion times for y_1 and y_2, respectively.*

Recall that a function g is said to be lower semicontinuous at a point y_0 if $\liminf_{y\to y_0} g(y)=g(y_0)$. Let $\|\cdot\|_{\mathbb{R}_+\times\mathbb{R}}$ denote the Euclidean norm on $\mathbb{R}_+\times\mathbb{R}$. The next fact about the monotone approximation of a lower semicontinuous function by Lipschitz continuous functions can be found in [6], Chapter XV, §4, Theorem 10.

LEMMA 4.2. *Suppose that $g\colon\mathbb{R}_+\times\mathbb{R}\to[0,\infty)$ is a lower semicontinuous and locally Lebesgue integrable function such that $g(t,x)>r>0$ for all $t\geqslant 0$ and $x\in\mathbb{R}$, where r is a constant. Then there exists a sequence of functions $\{g_n\}$, $n\geqslant 1$, such that*

i) *g_n is globally Lipschitz for all $n\geqslant 1$;*

ii) *$0<C\leqslant g_1\leqslant g_2\leqslant\ldots\leqslant g$, where C is a constant depending on r only;*

iii) *$g_n\uparrow g$ as $n\to\infty$ pointwise.*

The sequence $\{g_n\}$ can be chosen as

$$g_n(x)=\left(\inf_{y\in\mathbb{R}_+\times\mathbb{R}}\big(g(y)+n\|x-y\|_{\mathbb{R}_+\times\mathbb{R}}\big)-\frac{r}{2n}\right)\wedge n,\quad n\in\mathbb{N}. \tag{4.2}$$

The following lemma provides a generalization of classical Krylov's estimates to the case of symmetric stable processes with $1<\alpha\leqslant 2$.

LEMMA 4.3. *Let X be a solution of Eq.* (1.1), *where Z is a symmetric stable process of index $\alpha \in (1,2]$. Then, for any Borel measurable function $g\colon [0,\infty) \times \mathbb{R} \to [0,\infty]$ and all $t \geqslant 0$, $m \in \mathbb{N}$, and $x_0 \in \mathbb{R}$, there exists a constant N depending on α, m, and t only such that*

$$\mathbf{E}\int_0^{t\wedge\tau_m(X)} |b|^{\frac{\alpha}{2}}(s, x_0+X_s)g(s, x_0+X_s)\,\mathrm{d}s \leqslant N\|g\|_{2,t,m;l} := N\Big(\int_0^t\int_{-m}^m g^2(s,x)\,\mathrm{d}s\,\mathrm{d}x\Big)^{1/2}, \qquad (4.3)$$

where $\tau_m(X) = \inf\{t \geqslant 0\colon |x_0+X_t| \geqslant m\}$.

Estimate (4.3) was proved in [5] in the case of a Brownian motion and then generalized in [7] to the case of a symmetric stable process with $1 < \alpha < 2$.

Define the measure μ as

$$\mu(G) := \int_G |b|^{-\alpha}(y)l(\mathrm{d}y) \qquad (4.4)$$

for all $G \subseteq \mathbb{R}_+ \times \mathbb{R}$. By $\|\cdot\|_{2,t,m;\mu}$ we denote the norm defined in (4.3) with the Lebesgue measure l replaced by the measure μ. The following lemma follows from Corollary 33 in [8].

LEMMA 4.4. *Assume that b satisfies condition* (2) *and $\mathcal{M}_\alpha(b) = \mathcal{N}(b)$. Then there exists a sequence of lower semicontinuous and locally Lebesgue integrable functions $g_n \geqslant |b|^\alpha$, $n = 1,2\ldots$, with $\mathcal{N}(g_n) = \mathcal{N}(b)$ such that $\|g_n - |b|^\alpha\|_{2,t,m;\mu} \to 0$ as $n \to \infty$ for all $m \in \mathbb{N}$ and $t \geqslant 0$.*

Acknowledgment. The author thanks the anonymous referee for pointed out corrections and helpful remarks.

References

1. H.J. Engelbert and V.P. Kurenok, On one-dimensional stochastic equations driven by symmetric stable processes, in: R. Buckdahn, H.J. Engelbert, and M. Yor (Eds.), *Stochastic Processes and Related Topics* (2002), pp. 81–110.
2. H.J. Engelbert and W. Schmidt, Strong Markov continuous local martingales and solutions of one-dimensional stochastic differential equations, III, *Math. Nachr.*, **151**, 149–197 (1991).
3. J. Jacod, Calcul Stochastique et Problèmes de Martingales, *Lect. Notes Math.*, vol. 714, Springer-Verlag, Berlin (1979).
4. O. Kallenberg, *Foundations of Modern Probability*, Springer-Verlag, Berlin (1997).
5. N.V. Krylov, *Controlled Diffusion Processes*, Springer-Verlag, Berlin (1982).
6. I.P. Natanson, *Theory of Functions of a Real Variable*, Gostekhizdat, Moscow (1954) (in Russian).
7. H. Pragarauskas and P.A. Zanzotto, On one-dimensional stochastic differential equations driven by stable processes, *Lith. Math. J.*, **40**(1), 1–24 (2000).
8. P. Raupach, On driftless one-dimensional SDEs with time-dependent diffusion coefficients, *Stochastics Stochastics Rep.*, **67**, 207–230 (1998).

9. A. Rozkosz and L. Słomiński, On weak solutions of one-dimensional SDEs with time-dependent coefficients, *Stochastics Stochastics Rep.*, **42**, 199–208 (1993).
10. T. Senf, On one-dimensional stochastic differential equations without drift and with time-dependent diffusion coefficients, *Stochastics Stochastics Rep.*, **43**, 199–220 (1993).
11. P.A. Zanzotto, Representation of a class of semimartingales as stable integrals, *Theory Probab. Appl.*, **43**(4), 808–818 (1998).
12. P.A. Zanzotto, On stochastic differential equations driven by Cauchy process and the other stable Lévy motions, *Ann. Probab.*, **30**(2), 802–825 (2002).