

Stochastic equations with multidimensional drift driven by Levy processes

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Abstract

The stochastic equation $dX_t = dL_t + a(t, X_t)dt, t \geq 0$, is considered where L is a d -dimensional Levy process with the characteristic exponent $\psi(\xi), \xi \in \mathbb{R}, d \geq 1$. We prove the existence of (weak) solutions for a bounded, measurable coefficient a and any initial value $X_0 = x_0 \in \mathbb{R}^d$ when $(\operatorname{Re}\psi(\xi))^{-1} = o(|\xi|^{-1})$ as $|\xi| \rightarrow \infty$. The proof idea is based on Krylov's estimates for Levy processes with time-dependent drift and some variants of those estimates are derived in this note.

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1 Introduction

We consider the stochastic differential equation

$$dX_t = dL_t + a(t, X_t)dt, \quad t \geq 0, \quad X_0 = x_0 \in \mathbb{R}^d, \quad (1.1)$$

where $a : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a measurable drift coefficient and L is a d -dimensional Levy process with $L_0 = 0$ and the characteristic exponent $\psi(\xi), \xi \in \mathbb{R}^d, d \geq 1$. The goal is to prove the existence of a (weak) solution for any initial value $x_0 \in \mathbb{R}^d$.

In dealing with the equation (1.1) in terms of weak solutions, there are probably two major different approaches how to construct a solution. The one is to consider so-called *martingale problem* associated with the equation (1.1). This method relies essentially on working with analytic characteristics of processes satisfying the equation (1.1) rather than with stochastic processes themselves. The martingale method goes back to the pioneering work of Strook and Varadhan [17] where they treated under the assumption on continuity of the coefficient a the diffusion case of the equation (1.1), that is when L is a Brownian motion. In [16], one generalized those results to the case of an arbitrary Levy process L .

The weak solutions arise naturally in many applications such as stochastic control theory where one needs to construct a solution process X when the equation has discontinuous coefficients. Thus it is natural to investigate the existence problem in the case when the coefficient a doesn't need to be continuous. Several authors investigated that problem but the results obtained are covering, to our knowledge, only the time-independent case of (1.1), that is when $a(t, x) = a(x)$, and are dealing either with the particular form of the driven process L or the one-dimensional case of the equation. Thus, Tsuchiya [20] proved the existence of a solution for the time-independent coefficient a with the driven process L being a symmetric stable process of index $\alpha \in (1, 2)$. The case of time-independent equation (1.1) with arbitrary Levy process but one-dimensional state-space was considered by Tanaka, Tsuchiya, and Watanabe in [18] where they assumed a to be bounded and the process L to satisfy an additional condition. A similar problem but more in terms of the uniqueness in law of solutions was investigated by Komatsu [6], [7]. We mention also [19] where one proved the existence of a solution in the case of a Cauchy process when the coefficient a was assumed to be sufficiently small in the sup-norm in a neighbourhood of a constant function. In all papers mentioned above the proof techniques used were similar to the original method of Strook and Varadhan.

The second method of constructing a solution of the equation (1.1) is based on famous *embedding principle* of Skorokhod [15] which allows to work with approximating sequences of processes in terms of their weak convergence to the solution process. As an application of his embedding principle, Skorokhod proved the existence of solutions for the equation (1.1) with continuous coefficients (actually, for even more general equation with an arbitrary diffusion coefficient) when L is a Brownian motion. Using the Skorokhod's embedding principle, N.V. Krylov [8] was able later to prove the existence of solutions for the equation (1.1) for only bounded coefficient a . As the essential tool in his proof, Krylov used his integral estimates for processes of diffusion type. More precisely, let $f : [0, \infty) \times \mathbb{R}^d \rightarrow [0, \infty)$ be a measurable function and X be a stochastic process of the form

$$X_t = x_0 + \int_0^t b_s dL_s + \int_0^t a_s ds, \quad x_0 \in \mathbb{R}^d, \quad t \geq 0, \quad (1.2)$$

where (b_s) and (a_s) are two processes such that the corresponding stochastic and Lebesgue integrals are well-defined. When L is a Brownian motion, Krylov obtained, in particular, the estimates of the form

$$\mathbf{E} \int_0^\infty e^{-\phi_s} \Phi_s f(s, X_s) ds \leq N \|f\|_2, \quad (1.3)$$

where $\|f\|_2 := (\int_{[0,\infty) \times \mathbb{R}^d} |f|^2(s, y) ds dy)^{1/2}$ and ϕ, Φ are some predictable non-negative processes.

The Krylov's estimates turned to be very useful and important in many applications as well as in studying of stochastic differential equations themselves.

Using the Krylov's estimates approach, various authors were able to generalize the existence results of Krylov for the diffusion case. A. Roskosz and L. Slominski proved in [13],[14] that the equation (1.1) has a (non-exploding) solution when the coefficient a is assumed to be at most of linear growth. Furthermore, one proved in [10] that there is (possibly, exploding) solution of the equation (1.1) if one requires the coefficient a only to be locally integrable of the order $d + 1$.

There are known some generalizations of Krylov's estimates obtained for other driven processes L different from the Brownian motion process. We mention here the results of S. Anulova and H. Pragarauskas [2] who considered the case of diffusion processes with jumps, the situation when the Brownian motion was assumed to be a part of the driving process L . H. Pragarauskas studied the L_2 -estimates for purely jump Levy processes L being one-dimensional symmetric stable processes of index $\alpha \in (1, 2)$ when $a_s = 0$ [12]. We refer also to [9] where the case of one-dimensional symmetric stable processes with index $\alpha \in (1, 2)$ and $a_s \neq 0$ is discussed.

Unlike the martingale method and the Skorokhod embedding principle, S.I. Podolynny and N.I. Portenko [11] used a different idea to construct a solution of the time-independent equation (1.1) driven by a symmetric stable process of index $\alpha \in (1, 2)$. Their method was a purely analytical one and based on the fact that the transition probability function of the process X satisfies the backward Kolmogorov equation. The key idea was to obtain the corresponding estimates for the transition probability function of the solution process. Using those estimates they proved the existence of a solution when the drift coefficient a satisfies the assumption $a \in L_p(\mathbb{R})$ for $p > 1/(1 - \alpha)$.

Our approach to studying of the equation (1.1) is based on using the corresponding Krylov's estimates for processes of type (1.2) when $b = 1$. We shall prove here various L_2 -estimates for processes satisfying the equation (1.1). As an application of those estimates, the existence of a solution of the equation (1.1) for any initial value is verified. The coefficient a is assumed to be bounded and the Levy process to satisfy the condition

$$\frac{1}{(\mathcal{R}e\psi(\xi))} = o(|\xi|^{-1}) \quad \text{as} \quad |\xi| \rightarrow \infty. \quad (1.4)$$

The results obtained here generalize, in particular, those found by H. Tanaka, M. Tsuchiya, and S. Watanabe [18] for the case of one-dimensional, time-independent equation (1.1).

2 Preliminary facts

We begin with some definitions. By $\mathbb{ID}_{[0,\infty)}(\mathbb{R}^d)$ we denote the Skorokhod space, i.e. the set of all real-valued functions $x(\cdot) : [0, \infty) \rightarrow \mathbb{R}^d$ with right-continuous trajectories and with finite left limits. For simplicity, we shall write \mathbb{ID} instead of $\mathbb{ID}_{[0,\infty)}(\mathbb{R}^d)$. We will equip \mathbb{ID} with the σ -algebra \mathcal{D} generated by the Skorokhod topology. Under $\mathbb{ID}^n, n \geq 1$, we will understand the nd -dimensional Skorokhod space defined as $\mathbb{ID}^n = \mathbb{ID} \times \dots \times \mathbb{ID}$ with the corresponding σ -algebra \mathcal{D}^n being the direct product of n d -dimensional σ -algebras \mathcal{D} .

Let L be a process with $L_0 = 0$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and let $\mathbb{F} = (\mathcal{F}_t)$ be a filtration on $(\Omega, \mathcal{F}, \mathbf{P})$. We use the notation (L, \mathbb{F}) to express that L is adapted to the filtration \mathbb{F} . A process (L, \mathbb{F}) is said to be a d -dimensional *Levy process* if trajectories of L belong to \mathbb{ID} and

$$\mathbf{E} \left(e^{i\langle \xi, L_t - L_s \rangle} \middle| \mathcal{F}_s \right) = e^{-(t-s)\psi(\xi)}$$

for all $t > s \geq 0, \xi \in \mathbb{R}^d$ and a continuous function $\psi : \mathbb{R}^d \rightarrow \mathbf{C}$ where $\langle \cdot, \cdot \rangle$ denotes the dot product of two vectors in \mathbb{R}^d . The function ψ is called the *characteristic exponent* of the process L .

On another hand, there is one-to-one correspondence between the class of Levy processes and the class of infinitely divisible distributions. Let μ be a probability distribution on \mathbb{R}^d and

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{i\langle \xi, z \rangle} \mu(dz), \quad \xi \in \mathbb{R}^d,$$

defines the Fourier transform of μ . Then, μ is said to be infinitely divisible if, for every $n \geq 1$, there exists a probability distribution μ_n such that $\hat{\mu} = (\hat{\mu}_n)^n$. Moreover, the famous Levy-Khinchin formula says that μ is infinitely divisible distribution if and only if there is a continuous function $\psi : \mathbb{R}^d \rightarrow \mathbf{C}$ such that

$$\hat{\mu}(\xi) = e^{-\psi(\xi)}, \quad \xi \in \mathbb{R}^d,$$

and

$$\psi(\xi) = i\langle c, \xi \rangle + \frac{1}{2}\langle Q\xi, \xi \rangle + \int_{\mathbb{R}^d} \left(1 - e^{i\langle \xi, z \rangle} + i\langle \xi, z \rangle \mathbf{1}_{\{|z| < 1\}} \right) \nu(dz), \quad (2.1)$$

where $c \in \mathbb{R}^d$, Q is a positive definite, real-valued $d \times d$ matrix, and $\nu : \mathbb{R}^d \setminus \{0\} \rightarrow [0, \infty]$ is a Borel measure such that $\int (1 \wedge |z|^2) \nu(dz) < \infty$. Here $|\cdot|$

denotes the Euclidean norm of a vector in \mathbb{R}^d . One can easily show that if L is a Levy process then, for all $t \geq 0$,

$$\mathbb{E}e^{i\langle \xi, L_t \rangle} = e^{-t\psi(\xi)},$$

where ψ is the characteristic exponent of the corresponding infinitely divisible probability distribution that generates the process L .

We shall use the representation $\psi(\xi) = \mathcal{R}e\psi(\xi) + i\mathcal{I}m\psi(\xi)$ where the real valued functions $\mathcal{R}e\psi(\xi)$ and $\mathcal{I}m\psi(\xi)$ denote the real and imaginary part of $\psi(\xi)$, respectively. We remark also that $\psi(-\xi)$ is the characteristic exponent of the dual process $-L$ which coincides with the complex conjugate of the characteristic exponent $\psi(\xi)$ of the given process L . That is, $\psi(-\xi) = \overline{\psi(\xi)} = \mathcal{R}e\psi(\xi) - i\mathcal{I}m\psi(\xi)$.

The measure ν in the Levy-Khinchin formula is called the Levy measure and describes the intensity of jumps of L . In particular, if $\nu = 0$ and $c = 0$, then the Levy process L is a (standard) Brownian motion process. If $Q = 0$, then L is a purely jump Levy process. Because the equation (1.1) is well-studied in the case of Brownian motion, we shall restrict ourselves in this note to the case of purely discontinuous Levy processes ($Q = 0$). It follows then from (2.1) that

$$\mathcal{R}e\psi(\xi) = \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos \langle \xi, z \rangle) \nu(dz)$$

implying

$$\mathcal{R}e\psi(\xi) \geq 0 \quad \text{for all } \xi \in \mathbb{R}^d. \quad (2.2)$$

A stochastic process (X, \mathbb{F}) with trajectories in \mathbb{D} , defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, is called a (*weak*) *solution* of the equation (1.1) with initial value $x_0 \in \mathbb{R}^d$ if there exists a Levy process (L, \mathbb{F}) with a given characteristic exponent ψ such that

$$X_t = x_0 + L_t + \int_0^t a(s, X_s) ds, \quad t \geq 0 \quad \mathbf{P}\text{-a.s.}$$

We note that the last equality is understood componentwise.

Obviously, L is a Markov process as a process having independent increments. Hence it can be characterized in terms of Markov processes. For any function $f \in L^\infty(\mathbb{R}^d)$ and $t \geq 0$, define the operator

$$(P_t f)(x) := \int_{\Omega} f(x + L_t) d\mathbf{P}(\omega)$$

where $\mathcal{L}^\infty(\mathbb{R}^d)$ is the Banach space of functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ with the norm $\|f\|_\infty = \text{ess sup } |f(x)|$. The family $(P_t)_{t \geq 0}$ is called the family of convolution

operators associated with L . Formally, for a suitable class of functions $g : \mathbb{R}^d \rightarrow \mathbb{R}$, we can define so-called infinitesimal generator \mathcal{A} of the process L as

$$(\mathcal{A}g)(x) = \lim_{t \downarrow 0} \frac{(P_t g)(x) - g(x)}{t}.$$

It is well known that

$$(\mathcal{A}g)(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left(g(x+z) - g(x) - \mathbf{1}_{\{|z|<1\}} \langle \nabla g(x), z \rangle \right) \nu(z) dz$$

for any $g \in C^2$, where C^2 is the set of all bounded and twice continuously differentiable functions $g : \mathbb{R}^d \rightarrow \mathbb{R}$ and $\nabla g = (g_{x_1}, g_{x_2}, \dots, g_{x_d})$ is the gradient vector of g in $x = (x_1, x_2, \dots, x_d)$.

Notice finally that the use of Fourier transform can simplify calculations when working with infinitesimal operator \mathcal{A} . Let $g \in \mathcal{L}_1(\mathbb{R} \times \mathbb{R}^d)$ and

$$\hat{g}(\zeta, \xi) := \int_{\mathbb{R} \times \mathbb{R}^d} e^{is\zeta + i\langle x, \xi \rangle} g(s, x) ds dx$$

be the Fourier transform of g where $\zeta \in \mathbb{R}, \xi \in \mathbb{R}^d$. Clearly, according to the Fubini's theorem, the function $\hat{g}(\zeta, \xi)$ can be seen as the result of applying $d+1$ times one-dimensional Fourier transform to the function $g(s, x_1, \dots, x_d)$ in any order of permutation of variables (s, x_1, \dots, x_d) . The proof of the following facts is omitted because of being straightforward (cf. Proposition 2.1 in [9]).

Proposition 2.1 *Let \mathcal{A} be the infinitesimal generator of the Levy process L with the characteristic exponent ψ . We have:*

(i) *Assume that $g \in C^2(\mathbb{R}^d)$ and $\mathcal{A}g \in \mathcal{L}_1(\mathbb{R}^d)$. Then*

$$\widehat{(\mathcal{A}g)}(\xi) = -\psi(-\xi)\hat{g}(\xi).$$

(ii) *Let g be absolutely continuous on every compact subset of \mathbb{R}^d and $g_{x_j} \in \mathcal{L}_1(\mathbb{R}^d)$ where g_{x_j} is the partial derivative of g in the variable x_j for any $j = 1, \dots, d$. Then*

$$\widehat{g_{x_j}}(\xi) = -i\xi_j \hat{g}(\xi).$$

3 \mathcal{L}_2 -estimates

In this section we are going first to derive an L_2 -estimate for solutions of a given class of quasilinear partial differential equations. Based on that result, we shall

prove some Krylov's estimates for stable processes with drift that will be used later to prove the existence of solutions of the equation (1.1).

Let $K > 0$ be a constant and f be a nonnegative, measurable function such that $f \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d)$ where $C_0^\infty(\mathbb{R} \times \mathbb{R}^d)$ denotes the class of all infinitely many times differentiable realvalued functions with compact support defined on $\mathbb{R} \times \mathbb{R}^d$. Suppose further that L is a Levy process with the characteristic exponent $\psi(\xi)$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration \mathbb{F} . By \mathcal{T} we denote the class of all \mathbb{F} -predictable d -dimensional processes (a_t) such that $|a_t| \leq K$.

Consider the controlled processes X^a of the form

$$dX_t^a = dL_t + a_t dt$$

and, for any $\lambda > 0$, define the corresponding value function $v(t, x), (t, x) \in \mathbb{R} \times \mathbb{R}^d$, by

$$v(t, x) = \sup_{a \in \mathcal{T}} \mathbf{E} \int_0^\infty e^{-\lambda s} f(s + t, x + X_s^a) ds.$$

The Bellman principle of optimality can be formulated for the controlled process X^a and the value function v as follows:

For any \mathbb{F} -stopping time τ it holds

$$v(t, x) = \sup_{a \in \mathcal{T}} \mathbf{E} \left\{ \int_0^\tau e^{-\lambda s} f(s + t, x + X_s^a) ds + e^{-\lambda \tau} v(\tau + t, x + X_\tau^a) \right\}.$$

Using standard arguments, one can derive from the principle above the corresponding Bellman equation (a is a deterministic d -dimensional vector)

$$\sup_{|a| \leq K} \left\{ v_t(t, x) + \mathcal{A}v(t, x) - \lambda v(t, x) + \langle a, \nabla v \rangle(t, x) + f(t, x) \right\} = 0$$

which holds a.e. in $\mathbb{R} \times \mathbb{R}^d$. Here v_t denotes the partial derivative of the function $v(t, x)$ in t . It is easy to see that the Bellman equation is equivalent to the equation

$$v_t + \mathcal{A}v - \lambda v + K |\nabla v| + f = 0. \quad (3.1)$$

Now, for any measurable function $g : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, define

$$\|g\|_2 := \left(\int_{\mathbb{R} \times \mathbb{R}^d} h^2(t, x) dt dx \right)^{1/2}$$

to be the \mathcal{L}_2 -norm of g .

Lemma 3.1 For all $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, it holds

$$v(t, x) \leq N \|f\|_2, \quad (3.2)$$

where the constant N depends on K, ψ , and d only.

Proof. Let $q(t, x)$ be a nonnegative function such that $q \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d)$ and $\int_{\mathbb{R} \times \mathbb{R}^d} q(t, x) dt dx = 1$. For any function $g : \mathbb{R} \times \mathbb{R}^d \rightarrow [0, \infty)$ and any $\varepsilon > 0$ let

$$g^{(\varepsilon)}(t, x) = \frac{1}{\varepsilon^{d+1}} \int_{\mathbb{R} \times \mathbb{R}^d} q\left(\frac{t-s}{\varepsilon}, \frac{x-y}{\varepsilon}\right) g(s, y) ds dy$$

be the ε -convolution of g with q . By taking the ε -convolution on both sides of (3.1), we obtain

$$v_t^{(\varepsilon)} + \mathcal{A}v^{(\varepsilon)} - \lambda v^{(\varepsilon)} + K|\nabla v^{(\varepsilon)}| + f^{(\varepsilon)} = 0. \quad (3.3)$$

It follows from (3.3) that

$$\left(v_t^{(\varepsilon)} + \mathcal{A}v^{(\varepsilon)} - \lambda v^{(\varepsilon)}\right)^2 = \left(K|\nabla v^{(\varepsilon)}| + f^{(\varepsilon)}\right)^2$$

and

$$\begin{aligned} \|v_t^{(\varepsilon)} + \mathcal{A}v^{(\varepsilon)} - \lambda v^{(\varepsilon)}\|_2^2 &= \|(K|\nabla v^{(\varepsilon)}| + f^{(\varepsilon)})\|_2^2 \leq \\ &2K^2 \sum_{j=1}^d \|v_{x_j}^{(\varepsilon)}\|_{L_2}^2 + 2\|f^{(\varepsilon)}\|_2^2, \end{aligned} \quad (3.4)$$

where we have used the fact that $|\nabla v^{(\varepsilon)}|^2 = \sum_{j=1}^d \left(v_{x_j}^{(\varepsilon)}\right)^2$.

Now, applying Proposition 2.1, the Parseval identity and integration by parts to the last inequality, we can write in terms of Fourier transforms

$$\begin{aligned} \|J(\zeta, \xi, \lambda)|\hat{v}^{(\varepsilon)}\|_2^2 &\leq \\ 2K^2 \sum_{j=1}^d \|\xi_j |\hat{v}^{(\varepsilon)}\|_2^2 + 2\|\hat{f}^{(\varepsilon)}\|_2^2 &= 2K^2 \|\xi\| |\hat{v}^{(\varepsilon)}\|_2^2 + 2\|\hat{f}^{(\varepsilon)}\|_2^2, \end{aligned} \quad (3.5)$$

where

$$J(\zeta, \xi, \lambda) := \sqrt{[\operatorname{Re}\psi(\xi) + \lambda]^2 + [\zeta - \operatorname{Im}\psi(\xi)]^2}$$

and $|\xi|^2 = \xi_1^2 + \dots + \xi_d^2$.

Taking into account (1.4) and (2.2), we conclude that there exists a constant $\lambda_0 > 0$ such that

$$J(\zeta, \xi, \lambda_0) \geq \operatorname{Re}\psi(\xi) + \lambda_0 \geq 2K|\xi| \quad (3.6)$$

for all $(\zeta, \xi) \in \mathbb{R} \times \mathbb{R}^d$.

Combining the inequalities (3.5) and (3.6), we obtain for all $\lambda \geq \lambda_0$

$$\frac{1}{2} \|J(\zeta, \xi, \lambda) |\hat{v}^{(\varepsilon)}|\|_2^2 \leq 2 \| \|\hat{f}^{(\varepsilon)}\|_2^2. \quad (3.7)$$

Let

$$N_1 := \int_{\mathbb{R} \times \mathbb{R}^d} J(\zeta, \xi, \lambda)^{-2} d\zeta d\xi.$$

The constant N_1 depends on K and ψ only and is finite. Indeed,

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{R}^d} J(\zeta, \xi, \lambda)^{-2} d\zeta d\xi = \\ & \int_{\mathbb{R}^d} \left\{ \int_{\mathbb{R}} \frac{1}{[\mathcal{R}e\psi(\xi) + \lambda]^2 + [\zeta - \mathcal{I}m\psi(\xi)]^2} d\zeta \right\} d\xi = \\ & \pi \int_{\mathbb{R}^d} \frac{1}{|\lambda + \mathcal{R}e\psi(\xi)|} d\xi < \infty, \end{aligned}$$

the last inequality being true because of the assumption (1.4) and condition (2.2).

Using the estimate (3.7) and the inverse Fourier transform yields for all $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ and $\lambda \geq \lambda_0$

$$\begin{aligned} & \left(v^{(\varepsilon)}(t, x) \right)^2 \leq \\ & \frac{N_1}{4\pi^2} \|J(\zeta, \xi, \lambda) |\hat{v}^{(\varepsilon)}|\|_2^2 \leq \\ & \frac{N_1}{\pi^2} \| \|\hat{f}^{(\varepsilon)}\|_2^2 = \frac{N_1}{\pi^2} \| \|\hat{f}^{(\varepsilon)}\|_2^2. \end{aligned}$$

The result follows then by taking the limit $\varepsilon \rightarrow 0$ in the above inequality and using the Lebesgue dominated convergence theorem. \square

Remark 3.2 *Let L be a symmetric stable process of index $\alpha \in (1, 2)$. That is, $\psi(x) = |x|^\alpha$. It is then easy to see that the condition (1.4) is fulfilled.*

Now, let X be a solution of the equation (1.1) so that the assumption

$$|a(t, x)| \leq K \quad \text{for all } t \in \mathbb{R}, x \in \mathbb{R}^d \quad (3.8)$$

is satisfied. We are interested in \mathcal{L}_2 - estimates of the form

$$\mathbf{E} \int_0^\infty e^{-\lambda u} f(t_0 + u, x_0 + X_u) du \leq N \|f\|_2, \quad (3.9)$$

where $t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^d$.

Theorem 3.3 *Suppose X is a solution of the equation (1.1) and the assumptions (1.4) and (3.8) hold. Then, for any $t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^d, \lambda \geq \lambda_0$, and any measurable function $f : \mathbb{R} \times \mathbb{R}^d \rightarrow [0, \infty)$, it holds*

$$\mathbf{E} \int_0^\infty e^{-\lambda u} f(t_0 + u, x_0 + X_u) du \leq N \|f\|_2, \quad (3.10)$$

where the constant N depends on K and ψ only.

Proof. Assume first that $f \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d)$ so that there is a solution v of equation (3.1) satisfying the inequality (3.2). By taking the ε -convolution on both sides of (3.1), we obtain (3.3).

Then, for all $t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^d$, applying the Itô's formula to the expression

$$v^{(\varepsilon)}(t_0 + s, x_0 + X_s) e^{-\lambda s},$$

yields

$$\begin{aligned} & \mathbf{E} v^{(\varepsilon)}(t_0 + s, x_0 + X_s) e^{-\lambda s} - v^{(\varepsilon)}(t_0, x_0) = \\ & \mathbf{E} \int_0^s e^{-\lambda u} \left[\mathcal{A}v^{(\varepsilon)} - \lambda v^{(\varepsilon)} + \langle a(u, X_u), \nabla v^{(\varepsilon)} \rangle \right] (t_0 + u, x_0 + X_u) du \leq \\ & \mathbf{E} \int_0^s e^{-\lambda u} \left[\mathcal{A}v^{(\varepsilon)} - \lambda v^{(\varepsilon)} + K |\nabla v^{(\varepsilon)}| \right] (t_0 + u, x_0 + X_u) du \leq \\ & -\mathbf{E} \int_0^s e^{-\lambda u} f^{(\varepsilon)}(t_0 + u, x_0 + X_u) du. \end{aligned}$$

By Lemma 3.1

$$\mathbf{E} \int_0^s e^{-\lambda u} f^{(\varepsilon)}(t_0 + u, x_0 + X_u) du \leq \sup_{t_0, x_0} v^{(\varepsilon)}(t_0, x_0) \leq N \|f^{(\varepsilon)}\|_2.$$

It remains to pass to the limit in the above inequality letting $\varepsilon \rightarrow 0$ and $s \rightarrow \infty$ and to use the Fatou's lemma.

The inequality (3.10) can be extended in a standard way first to any function $f \in \mathcal{L}_2(\mathbb{R} \times \mathbb{R}^d)$ and then to any non-negative, measurable function using the monotone class theorem arguments (see, for example, [4], Theorem 20). \square

Corollary 3.4 *Suppose X is a solution of the equation (1.1) so that the assumptions (1.4) and (3.8) are fulfilled. Then, for any $t \geq 0$ and any measurable function $f : \mathbb{R} \times \mathbb{R}^d \rightarrow [0, \infty)$, it holds*

$$\mathbf{E} \int_0^t f(u, X_u) du \leq N \|f\|_2,$$

where the constant N depends on K, t , and ψ only.

Now, for arbitrary but fixed $t > 0, m \in \mathbb{N}$, define

$$\|f\|_{2,m,t} = \left(\int_0^t \int_{[-m,m]^d} |f(s,x)|^2 dx ds \right)^{\frac{1}{2}}$$

to be the \mathcal{L}_2 -norm of f on $[0,t] \times [-m,m]^d$ where $[-m,m]^d$ is the d -dimensional cube in \mathbb{R}^d with the side $[-m,m]$. Define the \mathbb{F} -stopping time $\tau_m(X)$ by

$$\tau_m(X) = \inf\{t \geq 0 : |X_t| > m\}.$$

Then, applying (3.10) to the function $\bar{f}(t,x) = f(t,x)\mathbf{1}_{[0,t] \times [-m,m]^d}(t,x)$, we obtain the following local version of Krylov's estimates

Corollary 3.5 *Let X be a solution of the equation (1.1) with the conditions (1.4) and (3.8) being satisfied. Then, for any $t \geq 0, m \in \mathbb{N}$, and any nonnegative measurable function f , it follows that*

$$\mathbf{E} \int_0^{t \wedge \tau_m(X)} f(u, X_u) du \leq N \|f\|_{2,m,t}, \quad (3.11)$$

where N is a constant depending on K, ψ, m , and t only.

4 Existence of solutions

Theorem 4.1 *Assume the assumptions (3.8) and (1.4) are true. Then, for any $x_0 \in \mathbb{R}^d$, there exists a solution of the equation (1.1).*

Proof. Because of a being bounded, we can find a sequence of functions $a_n(t,x), n \geq 1$ such that they are globally Lipschitz continuous and uniformly bounded by the constant K . Then, for any $n = 1, 2, \dots$, the equation (1.1) has a unique strong solution (see, for example, Theorem 9.1 in [5]). That is, for any fixed Levy process L with the characteristic exponent ψ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, there exists a sequence of processes $X^n, n = 1, 2, \dots$, such that

$$X_t^n = x_0 + L_t + \int_0^t a_n(s, X_s^n) ds, \quad t \geq 0, \quad \mathbf{P}\text{-a.s.} \quad (4.1)$$

Let

$$Y_t^n := \int_0^t a_n(s, X_s^n) ds$$

so that

$$X^n = x_0 + L + Y^n, n \geq 1.$$

Now we claim that the sequence of $3d$ -dimensional processes $Z^n := (X^n, Y^n, L)$, $n \geq 1$, is tight in the sense of weak convergence in $(\mathbb{D}^3, \mathcal{D}^3)$. Due to the Aldous' criterion ([1]), we have only to show that

$$\lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P} \left(\sup_{0 \leq s \leq t} |Z_s^n| > l \right) = 0 \quad (4.2)$$

for all $t \geq 0$ and

$$\limsup_{n \rightarrow \infty} \mathbf{P} \left(|Z_{t \wedge (\tau^n + r_n)}^n - Z_{t \wedge \tau^n}^n| > \varepsilon \right) = 0$$

for all $t \geq 0, \varepsilon > 0$, every sequence of \mathbb{F} -stopping times τ^n , and every sequence of real numbers r_n such that $r_n \downarrow 0$.

It is obvious that both conditions are satisfied because of the uniform boundedness of the coefficients $a_n, n \geq 1$.

Since the sequence $\{Z^n\}$ is tight, there exists a subsequence $\{n_k\}, k = 1, 2, \dots$, a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ and the process \tilde{Z} on it with values in $(\mathbb{D}^3, \mathcal{D}^3)$ such that Z^{n_k} converges weakly (in distribution) to the process \tilde{Z} as $k \rightarrow \infty$. For simplicity, let $\{n_k\} = \{n\}$.

According to the embedding principle of Skorokhod (see, e.g. Theorem 2.7 in [5]), there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ and the processes $\tilde{Z} = (\tilde{X}, \tilde{Y}, \tilde{S}), \tilde{Z}^n = (\tilde{X}^n, \tilde{Y}^n, \tilde{S}^n), n = 1, 2, \dots$, on it such that

- i) $\tilde{Z}^n \rightarrow \tilde{Z}$ as $n \rightarrow \infty$ $\tilde{\mathbf{P}}$ -a.s.
- ii) $\tilde{Z}^n = Z^n$ in distribution for all $n = 1, 2, \dots$

Using standard measurability arguments ([8], chapter 2), one can prove that the processes \tilde{L}^n and \tilde{L} are Levy processes with the characteristic exponent ψ with respect to the augmented filtrations $\tilde{\mathbb{F}}^n$ and $\tilde{\mathbb{F}}$ generated by processes \tilde{Z}^n and \tilde{Z} , respectively.

Using the properties *i*) and *ii*), and the equation (4.1), one can show (cf. [8], chapter 2) that

$$\tilde{X}_t^n = x_0 + \tilde{L}_t^n + \int_0^t a_n(s, \tilde{X}_s^n) ds, \quad t \geq 0, \quad \tilde{\mathbf{P}}\text{-a.s.}$$

On the other hand, the properties *i*), *ii*) and the quasi-left continuity of the the processes \tilde{X}^n yield

$$\lim_{n \rightarrow \infty} \tilde{X}_t^n = \tilde{X}_t, \quad t \geq 0, \quad \tilde{\mathbf{P}}\text{-a.s.} \quad (4.3)$$

Therefore, in order to show that the process \tilde{X} is a solution of the equation (1.1), it suffices to verify that, for all $t \geq 0$,

$$\lim_{n \rightarrow \infty} \int_0^t a_n(s, \tilde{X}_s^n) ds = \int_0^t a(s, \tilde{X}_s) ds \quad \tilde{\mathbf{P}}\text{-a.s.} \quad (4.4)$$

The following fact can be proven similar as Lemma 4.2 in [9].

Lemma 4.2 For any Borel measurable function $f : \mathbb{R} \times \mathbb{R}^d \rightarrow [0, \infty)$ and any $t \geq 0$, there exists a sequence $m_k \in (0, \infty)$, $k = 1, 2, \dots$ such that $m_k \uparrow \infty$ as $k \rightarrow \infty$ and it holds

$$\tilde{\mathbf{E}} \int_0^{t \wedge \tau_{m_k}(\tilde{X})} f(s, \tilde{X}_s) ds \leq N \|f\|_{2, m_k, t},$$

where the constant N depends on K, ψ, t and m_k only.

Without loss of generality, we can assume in the lemma above that $\{m_k\} = \{m\}$. Now, to prove (4.4), it is enough to verify that for all $t \geq 0$ and $\varepsilon > 0$ we have

$$\lim_{n \rightarrow \infty} \tilde{\mathbf{P}} \left(\left| \int_0^t a_n(s, \tilde{X}_s^n) ds - \int_0^t a(s, \tilde{X}_s) ds \right| > \varepsilon \right) = 0. \quad (4.5)$$

In order to prove (4.5) we estimate for a fixed $k \in \mathbb{N}$

$$\begin{aligned} & \tilde{\mathbf{P}} \left(\left| \int_0^t a_n(s, \tilde{X}_{s-}^n) ds - \int_0^t a(s, \tilde{X}_{s-}) ds \right| > \varepsilon \right) \leq \\ & \tilde{\mathbf{P}} \left(\left| \int_0^t a_k(s, \tilde{X}_{s-}^n) ds - \int_0^t a_k(s, \tilde{X}_{s-}) ds \right| > \frac{\varepsilon}{3} \right) \\ & + \tilde{\mathbf{P}} \left(\left| \int_0^{t \wedge \tau_m(\tilde{X}^n)} [a_k - a_n](s, \tilde{X}_{s-}^n) ds \right| > \frac{\varepsilon}{3} \right) \\ & + \tilde{\mathbf{P}} \left(\left| \int_0^{t \wedge \tau_m(\tilde{X})} [a_k - a](s, \tilde{X}_{s-}) ds \right| > \frac{\varepsilon}{3} \right) + \tilde{\mathbf{P}} \left(\tau_m(\tilde{X}^n) < t \right) + \tilde{\mathbf{P}} \left(\tau_m(\tilde{X}) < t \right) = \\ & \Delta_{n,k}^1 + \Delta_{n,k,m}^2 + \Delta_{k,m}^3 + \tilde{\mathbf{P}} \left(\tau_m(\tilde{X}^n) < t \right) + \tilde{\mathbf{P}} \left(\tau_m(\tilde{X}) < t \right). \end{aligned}$$

By Chebyshev's inequality and Lebesgue bounded convergence theorem, $\Delta_{n,k}^1 \rightarrow 0$ as $n \rightarrow \infty$. To show that $\Delta_{n,k,m}^2 \rightarrow 0$ as $n \rightarrow \infty$ and $\Delta_{k,m}^3 \rightarrow 0$ as $k \rightarrow \infty$, we use first the Chebyshev's inequality and then Corollary 3.5 and Lemma 4.2, respectively, to estimate

$$\Delta_{n,k,m}^2 \leq \frac{3}{\varepsilon} N \|a_k - a_n\|_{2, m, t} \quad (4.6)$$

and

$$\Delta_{k,m}^3 \leq \frac{3}{\varepsilon} N \|a_k - a\|_{2, m, t} \quad (4.7)$$

where the constant N depends on K, m, t , and ψ only. Obviously, $\|a_n - a\|_{2, m, t} \rightarrow 0$ as $n \rightarrow \infty$ implying that the right-hand sides in (4.6) and (4.7) converge to 0 by letting first $n \rightarrow \infty$ and then $k \rightarrow \infty$.

Because of the property $\tau_m(\tilde{X}^n) \rightarrow \tau_m(\tilde{X})$ as $n \rightarrow \infty$ $\tilde{\mathbf{P}}$ -a.s.,

$$\tilde{\mathbf{P}} \left(\tau_m(\tilde{X}^n) < t \right) \rightarrow \tilde{\mathbf{P}} \left(\tau_m(\tilde{X}) < t \right) \quad \text{as } n \rightarrow \infty$$

for all $m \in \mathbb{N}, t > 0$. Therefore, the last two terms can be made arbitrarily small by choosing large enough m for all n due to the fact that the sequence of processes \tilde{X}^n satisfies the property (4.2). This proves (4.5). Hence \tilde{X} is a solution of the equation (1.1). \square

Taking into account Remark 3.2 amounts us to state the following

Corollary 4.3 *Let L be a symmetric stable process of index $\alpha \in (1, 2)$ and $a(t, x)$ be measurable and bounded. Then, for any initial value $x_0 \in \mathbb{R}^d$, there exists a solution of the equation (1.1).*

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