

Time change method and SDEs with nonnegative drift

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Abstract

Using the time change method we show how to construct a solution of the stochastic equation $dX_t = b(X_{t-})dZ_t + a(X_t)dt$ with a nonnegative drift a provided there exists a solution of the auxiliary equation $dL_t = [a^{-1/\alpha}b](L_{t-})d\bar{Z}_t + dt$ where Z, \bar{Z} are two symmetric stable processes of the same index $\alpha \in (0, 2]$. The approach allows us to prove the existence of solutions for both stochastic equations for the values $0 < \alpha < 1$ and only measurable coefficients a and b satisfying some conditions of boundedness. The existence proof for the auxiliary equation uses the method of integral estimates in the sense of Krylov.

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1 Introduction

The use of time change method in constructing of solutions of one-dimensional Itó equations is well-known. Usually, if the equation involves the drift term, one has then to apply the time change method in conjunction with a particular space transformation (drift transformation).

Here we shall use the time change method to construct a solution of the equation

$$dX_t = b(X_{t-})dZ_t + a(X_t)dt, \quad t \geq 0, X_0 = x_0 \in \mathbb{R}, \quad (1.1)$$

where Z is a symmetric stable process of index $\alpha \in (0, 2]$. The coefficients a and b are assumed to be only Borel measurable satisfying some boundedness conditions.

The equation (1.1) without drift ($a = 0$) is well studied. The case of $\alpha = 2$ was treated in detail in series of papers by H.J. Engelbert and W. Schmidt in 80th. They were able to find sufficient and necessary conditions for existence of solutions. We refer here, for example, to [6] and [8]. The general case with arbitrary $\alpha \in (0, 2]$ but still with $a = 0$ was studied by P. Zanzotto in [19] and [20] who, in particular, generalized the results of Engelbert and Schmidt for $\alpha \in (1, 2]$. The main method used for SDEs (1.1) without drift was the time change method.

The equation (1.1) with drift and $\alpha = 2$ was studied by H.J. Engelbert and W. Schmidt in [7] where one proved the existence of solutions under very general assumptions on the coefficients a and b combining the time change method and the method of drift transformation due to A. Zvonkin.

The case of equation (1.1) with $\alpha \in (1, 2)$ was considered in [11]. In particular, one had shown in [11] how one can obtain a solution X of (1.1) for any $\alpha \in (0, 2]$ by time change method if one has a process Y satisfying the equation

$$dY_t = d\tilde{Z}_t + c(Y_t)dt, \quad t \geq 0, \quad Y_0 = 0, \quad (1.2)$$

where $c = a|b|^{-\alpha}$ with $|b|^{-\alpha} := 1/|b|^\alpha$ and \tilde{Z} is a symmetric stable process of the same index α . In order to solve the equation (1.2), one used the method of so-called Krylov's estimates for processes Y . The tools developed there worked only for $\alpha \in (1, 2)$ but *not for the case of $\alpha \leq 1$* . Some results for equation (1.1) with $\alpha = 1$ and measurable coefficients were obtained in [12]. To our knowledge, there are no existence results known for equation (1.1) with $\alpha < 1$ in the case of only measurable coefficients.

The purpose of this paper is twofold. From one side, we suggest a method how to construct a process Y satisfying the equation (1.2) applicable for all values of $\alpha \in (0, 2]$. More precisely, we consider the auxiliary equation

$$dL_t = [a^{-1/\alpha}b](L_{t-})d\bar{Z}_t + dt, \quad t \geq 0, \quad L_0 = 0, \quad (1.3)$$

where \bar{Z} is a symmetric stable process of the same index α and $a^{-1/\alpha} := 1/a^{1/\alpha}$. Providing there is a solution L of the equation (1.3), a suitable time change in the process L will lead to a solution Y of the equation (1.2). However, in order for the time change to work, one has to require the drift coefficient a to be *nonnegative*. To prove the existence of solutions of equation (1.3), we shall use the method of Krylov's estimates for processes L . Conceptually, to obtain the corresponding integral estimates, we follow the idea similar to that used in [11] for processes Y and X . From another side, the usage of the method based on the equation (1.3) allows us to prove the existence of solutions of equation (1.1) with $\alpha < 1$ and only measurable coefficients a and b satisfying some assumptions of boundedness.

It should be noticed that the existence of solutions of equation (1.1) in the case of $0 < \alpha < 1$ and $a = 0$ with measurable coefficient b was proven in [19]. One assumed there the coefficient b to satisfy some additional assumption of local integrability. The main novelty of the results obtained here to compare with those in [19] is that one has the presence of the drift term a . Moreover, the handling of the equation (1.1) with drift seems to be more complicated and requires different approaches as for the equation (1.1) in Brownian motion case ($\alpha = 2$).

The introduction would be incomplete without mentioning of the results known for the equation (1.1) with $b = 1$ (similarly, for (1.2) with $c = a$). Thus, in [18] one studied the solutions under some conditions different for cases $0 < \alpha < 1$, $\alpha = 1$, and $1 < \alpha < 2$. Without going into details, we only mention that one required in the case of $\alpha < 1$ the coefficient a to satisfy some properties of smoothness. Moreover, the method used in [18] was a purely analytical one based on properties of corresponding Markov processes. More recently, N. Portenko [14] proved the existence of solutions of equation (1.1) with $b = 1$ and $\alpha \in (1, 2)$ under assumption $|a|^p \in L(\mathbb{R})$ for $p > 1/(\alpha - 1)$ where he used his own estimates for transition probability density function of the solution process.

2 Preliminaries

We shall denote by $\mathbf{D}_{[0, \infty)}(\mathbb{R})$ the Skorokhod space, i.e. the set of all real valued functions $z : [0, \infty) \rightarrow \mathbb{R}$ with right-continuous trajectories and with finite left limits (also called *càdlàg* functions). For simplicity, we shall write \mathbf{D} instead of $\mathbf{D}_{[0, \infty)}(\mathbb{R})$. We will equip \mathbf{D} with the σ -algebra \mathcal{D} generated by the Skorokhod topology. Under \mathbf{D}^n we will understand the n -dimensional Skorokhod space defined as $\mathbf{D}^n = \mathbf{D} \times \dots \times \mathbf{D}$ with the corresponding σ -algebra \mathcal{D}^n being the direct product of n one-dimensional σ -algebras \mathcal{D} .

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space carrying a process Z with $Z_0 = 0$ and let $\mathbb{F} = (\mathcal{F}_t)$ be a filtration on $(\Omega, \mathcal{F}, \mathbf{P})$. The notation (Z, \mathbb{F}) means that Z is adapted to the filtration \mathbb{F} . We call (Z, \mathbb{F}) a symmetric stable process of index $\alpha \in (0, 2]$ if trajectories of Z belong to \mathbf{D} and

$$\mathbf{E} \left(e^{i\xi(Z_t - Z_s)} | \mathcal{F}_s \right) = e^{-(t-s)|\xi|^\alpha}$$

for all $t > s \geq 0$ and $\xi \in \mathbb{R}$. If $\alpha = 2$, then $Z = W$ is a process of Brownian motion with the variance $2t$. For $\alpha = 1$ we have a Cauchy process with unbounded second moment. In general, $\mathbf{E}|Z_t|^\beta < \infty$ for $\beta < \alpha$. It is well-known that the process of Brownian motion W is the only symmetric stable process with continuous paths.

If $\alpha \in (0, 2)$, one has the following quasi-isometrical property proven by J. Rosinski and W. Woyczynski [17]: there exist constants c_α and C_α depending

on α only such that for all $t > 0$

$$c_\alpha \mathbf{E} \int_0^t |f_s|^\alpha ds \leq \sup_{\lambda > 0} \lambda^\alpha \mathbf{P} \left(\sup_{s \leq t} \left| \int_0^s f_u dZ_u \right| > \lambda \right) \leq C_\alpha \mathbf{E} \int_0^t |f_s|^\alpha ds. \quad (2.1)$$

For all $0 < \alpha \leq 2$, Z is a Markov process and can be characterized in terms of analytic characteristics of Markov processes. First, for any function $f \in L^\infty(\mathbb{R})$ and $t \geq 0$, we can define the operator

$$(P_t f)(x) := \int_{\Omega} f(x + Z_t) d\mathbf{P}(\omega)$$

where $L^\infty(\mathbb{R})$ is the Banach space of functions $f : \mathbb{R} \rightarrow \mathbb{R}$ with the norm $\|f\|_\infty = \text{ess sup } |f(x)|$. The family $(P_t)_{t \geq 0}$ is called the family of convolution operators associated with Z . Formally, for a suitable class of functions $g(x)$, let

$$(\mathcal{L}g)(x) = \lim_{t \downarrow 0} \frac{(P_t g)(x) - g(x)}{t}$$

called the infinitesimal generator of the process Z .

It is known that for $\alpha < 2$

$$(\mathcal{L}g)(x) = \int_{\mathbb{R} \setminus \{0\}} [g(x+z) - g(x) - \mathbf{1}_{\{|z| < 1\}} g'(x)z] \frac{k_\alpha}{|z|^{1+\alpha}} dz \quad (2.2)$$

for any $g \in C^2$, where C^2 is the set of all bounded and twice continuously differentiable functions $g : \mathbb{R} \rightarrow \mathbb{R}$ and k_α is a suitable constant. In contrary to the case of $\alpha \in (0, 2)$, the infinitesimal generator of a Brownian motion process ($\alpha = 2$) is the Laplacian, that is, the second derivative operator.

We notice also that the use of Fourier transform can simplify calculations when working with infinitesimal generator \mathcal{L} . Let $g \in L_1(\mathbb{R})$ and

$$\hat{g}(\xi) := \int_{\mathbb{R}} e^{iz\xi} g(z) dz$$

be the Fourier transform of g . The following fact will be used later (cf. Proposition 2.1 in [11]).

Proposition 2.1 *Let \mathcal{L} be the infinitesimal generator of a symmetric stable process Z . Assume that $g \in C^2(\mathbb{R})$ and $\mathcal{L}g \in L_1(\mathbb{R})$. Then*

$$\widehat{(\mathcal{L}g)}(\xi) = -|\xi|^\alpha \hat{g}(\xi).$$

The existence of solutions of stochastic equations (1.1)-(1.3) is understood here in the weak sense. For instance, we say that equation (1.1) has a solution

if there exist a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a filtration \mathbb{F} and processes X and Z on it such that it holds

$$X_t = x_0 + \int_0^t b(X_{s-})dZ_s + \int_0^t a(X_s)ds, \quad t \geq 0 \quad \mathbf{P} - \text{a.s.} \quad (2.3)$$

where (Z, \mathbb{F}) is a symmetric stable process of given index α . The definitions for equations (1.2) and (1.3) are similar.

3 Time change method

Here we are going to show how to construct a solution of equation (1.1) for any $\alpha \in (0, 2]$ using the time change method and equations (1.2) and (1.3). The method of time change is well-known in the theory of stochastic processes but plays also an important role in many applications, including the area of mathematical finance. There is an excessive application literature; we refer to [2], [3] only with many other references therein.

Recall first that a process A is called a \mathbb{F} -time change if it is an increasing, right-continuous process with $A_0 = 0$ such that A_t is a \mathbb{F} -stopping time for any $t \geq 0$ (cf. [9], chapter 4). Define $T_t =: \inf\{s \geq 0 : A_s > t\}$ called the right-continuous inverse process to A . By definition, T is an increasing process starting at zero. It is easy to see that T is a \mathbb{F} -adapted process if and only if A is a \mathbb{F} -time change.

Proposition 3.1 *Let $\alpha \in (0, 2]$ and assume that there exist constants $\delta_1 > 0$ and $\delta_2 > 0$ such that $\delta_1 \leq |b| \leq \delta_2$. Then, for any initial value $x_0 \in \mathbb{R}$, the equation (1.1) has a solution if and only if the equation (1.2) has a solution.*

Proof. Suppose first that X is a solution of the equation (1.1) which means that the equation (2.3) is satisfied. The integrals on the right-hand side of (2.3) are well-defined and are \mathbf{P} -a.s. finite for all $t \geq 0$. Let

$$A_t = \int_0^t |b|^\alpha(X_s)ds$$

and

$$T_t = \inf\{s \geq 0 : A_s > t\}.$$

It can be easily verified that the process T satisfies the relation

$$T_t = \int_0^t |b|^{-\alpha}(X_{T_s})ds.$$

By definition, the process A is \mathbb{F} -adapted so that its right-inverse process T is a \mathbb{F} -time change process defined for all $t \geq 0$. We notice that (T_t) is a global

time change* because $A_\infty = \lim_{t \uparrow \infty} A_t = \infty$. Now define

$$Y_t = X_{T_t}, \quad \mathcal{G}_t = \mathcal{F}_{T_t}.$$

Applying the time change $t \rightarrow T_t$ to the semimartingale X in (2.3) (see [10], Chapter 10) and using the change of variables rule in Lebesgue-Stieltjes integral (see ch. 0, (4.9) in [16]) yields

$$Y_t = x_0 + \int_0^{T_t} b(X_{s-}) dZ_s + \int_0^t a(Y_s) dT_s.$$

It remains to notice that the process

$$\tilde{Z}_t := \int_0^{T_t} b(X_{s-}) dZ_s$$

is nothing but a symmetric stable process of the index α (see [17], Theorem 3.1). Hence Y is a solution of the equation (1.2).

The proof of the opposite direction is a very similar one. For this, suppose that the process Y is a solution of the equation (1.2) defined on a probability space $(\Omega, \mathcal{G}, \mathbf{P})$ with a filtration \mathbb{G} where \tilde{Z} is a symmetric stable process adapted to \mathbb{G} . Define

$$T_t = \int_0^t |b|^{-\alpha}(Y_s) ds$$

and let

$$X_t = Y_{A_t}, \quad \mathcal{F}_t = \mathcal{G}_{A_t}$$

for all $t \geq 0$ where A is the right inverse to T and $T_\infty = \lim_{t \uparrow \infty} T_t = \infty$. By applying the global time change $t \rightarrow A_t$ to the semimartingale Y in (1.2) we obtain

$$\tilde{Z}_{A_t} = X_t - x_0 - \int_0^t a(X_s) ds.$$

Using simple time change arguments (cf. [5]), we can conclude that there exists a symmetric stable process Z defined on the same probability space such that

$$\tilde{Z}_{A_t} = \int_0^t b(X_{s-}) dZ_s. \quad (3.1)$$

This proves that X is a solution of the equation (1.1). \square

Proposition 3.2 *Additionally to assumptions of Proposition 3.1, suppose that there exist strictly positive constants K_1 and K_2 such that $K_1 \leq a(x) \leq K_2$ for all $x \in \mathbb{R}$. Then, for any $x_0 \in \mathbb{R}$, the equation (1.2) has a solution if and only if the equation (1.3) has a solution.*

*That is, $T_t \in [0, \infty)$ for all $t \geq 0$.

Proof. Let (L, \mathbb{F}) be a solution of equation (1.3) defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ with a filtration $\tilde{\mathbb{F}}$. Set

$$\tau_t := \int_0^t a^{-1}|b|^\alpha(L_s)ds$$

and define by τ^{-1} the right-inverse process to τ . It is easy to see that τ^{-1} is a strictly increasing, continuous $\tilde{\mathbb{F}}$ -time change. Moreover, it is a global time change because of the assumptions of the Proposition. It can be directly verified that

$$\tau_t^{-1} = \int_0^t a|b|^{-\alpha}(L_{\tau_s^{-1}})ds.$$

Now, letting $Y_t = L_{\tau_t^{-1}}$ and applying the time change $t \rightarrow \tau_t^{-1}$ to the relation (1.3) yields

$$Y_t = \int_0^{\tau_t^{-1}} [a^{-1/\alpha}b](L_{s-})d\bar{Z}_s + \tau_t^{-1} = \int_0^{\tau_t^{-1}} [a^{-1/\alpha}b](L_{s-})d\bar{Z}_s + \int_0^t [a|b|^{-\alpha}](Y_s)ds.$$

It remains to notice that the first integral on the right-hand side of last relation is nothing but a symmetric stable process \tilde{Z} of the same index α (cf. [17]) proving that Y is a solution of equation (1.2).

On another hand, assuming that Y is a solution of equation (1.2) defined of a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ with a filtration $\tilde{\mathbb{F}}$, we let

$$T_t = \int_0^t [a|b|^{-\alpha}](Y_s)ds$$

so that the right-inverse process T^{-1} has the form

$$T_t^{-1} = \int_0^t [a^{-1}|b|^\alpha](L_s)ds$$

where $L_t := Y_{T_t^{-1}}$.

After the time change $t \rightarrow T_t^{-1}$ in (1.2), we obtain

$$L_t = \tilde{Z}_{T_t^{-1}} + t.$$

Once again, by standard time change arguments (cf. [5]), there is a symmetric stable process \tilde{Z} of the same index α such that

$$\tilde{Z}_{T_t^{-1}} = \int_0^t [a^{-1/\alpha}b](L_{t-})d\bar{Z}_t.$$

Hence L satisfies equation (1.3). \square

Remark 3.3 *In Propositions 3.1 and 3.2 we required the coefficients a and b to be bounded from above and "away from zero". This allowed for simple time change proofs and lead to so-called nonexploding solutions for equations (1.1)-(1.3). Those assumptions on a and b could be relaxed in the sense allowing solutions to have explosions. However, in this note we do not consider the case of exploding solutions.*

4 Some integral estimates

Let K be a strictly positive constant and Z be a symmetric stable process of index $0 < \alpha < 1$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration \mathbf{F} . By \mathcal{I} we denote the class of all \mathbf{F} -predictable one-dimensional processes γ_t such that $|\gamma_t|^\alpha \leq K$.

For any $x \in \mathbb{R}$, $\lambda > 0$, and any nonnegative, measurable function $f \in C_0^\infty(\mathbb{R})^\dagger$ define the value function $v(x)$ as

$$v(x) = \sup_{\gamma \in \mathcal{I}} \mathbf{E} \int_0^\infty e^{-\lambda s} f(x + X_s^\gamma) ds,$$

where the process X^γ is given by

$$dX_t^\gamma = \gamma_t dZ_t + dt.$$

Then, for the value function v and the process X^γ , the Bellman principle of optimality can be formulated as follows (cf. [13]): for any $[0, \infty)$ -valued \mathbf{F} -stopping time τ it holds

$$v(x) = \sup_{\gamma \in \mathcal{I}} \mathbf{E} \left\{ \int_0^\tau e^{-\lambda s} f(x + X_s^\gamma) ds + e^{-\lambda \tau} v(x + X_\tau^\gamma) \right\}.$$

Using standard arguments, one can derive from the principle above the corresponding Bellman equation (γ is deterministic)

$$\sup_{|\gamma|^\alpha \leq K} \left\{ |\gamma|^\alpha \mathcal{L}v(x) - \lambda v(x) + v_x(x) + f(x) \right\} = 0 \quad (4.1)$$

which holds a.e. in \mathbb{R} .

Define $A = \{x : \mathcal{L}v(x) > 0\}$. Then, the Bellman equation is equivalent to two equations

$$\begin{cases} K \mathcal{L}v - \lambda v + v_x + f = 0 & \text{on } A \\ -\lambda v + v_x + f = 0 & \text{on } A^c. \end{cases} \quad (4.2)$$

Lemma 4.1 *For all $x \in \mathbb{R}$, it holds*

$$v(x) \leq N \|f\|_2 := N \left(\int_{\mathbb{R}} f^2(y) dy \right)^{1/2}, \quad (4.3)$$

where the constant N depends on K and α only.

[†] $C_0^\infty(\mathbb{R})$ denotes the class of all infinitely many times differentiable real valued functions with compact support defined on \mathbb{R}

Proof. For any function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that $h \in L_1(\mathbb{R})$ and any $\varepsilon > 0$ we define

$$h^{(\varepsilon)}(x) = \frac{1}{\varepsilon} \int_{\mathbb{R}} h(y) q\left(\frac{x-y}{\varepsilon}\right) dy$$

to be the ε -convolution of h with a smooth function q such that $q \in C_0^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} q(x) dx = 1$.

For $\varepsilon > 0$, let

$$f^{(\varepsilon)} := \begin{cases} \lambda v^{(\varepsilon)} - K\mathcal{L}v^{(\varepsilon)} - v_x^{(\varepsilon)} & \text{on } A \\ \lambda v^{(\varepsilon)} - v_x^{(\varepsilon)} & \text{on } A^c. \end{cases} \quad (4.4)$$

It follows that

$$K\mathcal{L}v^{(\varepsilon)} - \lambda v^{(\varepsilon)} + v_x^{(\varepsilon)} = -f + K\mathcal{L}v^{(\varepsilon)} \mathbf{1}_{A^c}$$

so that

$$\left(K\mathcal{L}v^{(\varepsilon)} - \lambda v^{(\varepsilon)} + v_x^{(\varepsilon)}\right)^2 \leq 2f^{(\varepsilon)^2} + 2K^2 \left(\mathcal{L}v^{(\varepsilon)}\right)^2.$$

Obviously, $f^{(\varepsilon)}$ is square integrable and (4.2) implies that $f^{(\varepsilon)} \rightarrow f$ as $\varepsilon \downarrow 0$ a.s. in \mathbb{R} .

Now, applying Proposition 2.1, the Parseval identity and integration by parts to the inequality

$$\begin{aligned} & \int_{\mathbb{R}} \left(K\mathcal{L}v^{(\varepsilon)} - \lambda v^{(\varepsilon)} + v_x^{(\varepsilon)}\right)^2 dx \leq \\ & 2 \int_{\mathbb{R}} \left((f^{(\varepsilon)})^2(x) dx + 2K^2 \int_{\mathbb{R}} (\mathcal{L}v^{(\varepsilon)})^2(x) dx \right) \end{aligned}$$

yields

$$\begin{aligned} & \int_{\mathbb{R}} |\hat{v}^{(\varepsilon)}(\xi)|^2 \left([K|\xi|^\alpha + \lambda]^2 + |\xi|^2\right) d\xi \leq \\ & 2 \int_{\mathbb{R}} \left(|\hat{f}^{(\varepsilon)}(\xi)|^2 d\xi + 2K^2 \int_{\mathbb{R}} |\xi|^{2\alpha} |\hat{v}^{(\varepsilon)}(\xi)|^2 d\xi. \right) \end{aligned} \quad (4.5)$$

One sees easily that there exists a constant $\lambda_0 > 0$ such that

$$[K|\xi|^\alpha + \lambda_0]^2 + |\xi|^2 \geq 4K^2 |\xi|^{2\alpha} \quad (4.6)$$

for all $\xi \in \mathbb{R}$.

Combining the inequalities (4.5) and (4.6), we obtain for all $\lambda \geq \lambda_0$

$$\frac{1}{2} \int_{\mathbb{R}} |\hat{v}^{(\varepsilon)}(\xi)|^2 \left([K|\xi|^\alpha + \lambda]^2 + |\xi|^2\right) d\xi \leq 2 \int_{\mathbb{R}} |\hat{f}^{(\varepsilon)}(\xi)|^2 d\xi. \quad (4.7)$$

Let

$$N_1 := \int_{\mathbb{R}} \frac{d\xi}{[K|\xi|^\alpha + \lambda]^2 + |\xi|^2}.$$

Clearly, the constant N_1 is finite and depends on K and α only.

Using the estimate (4.7) and the inverse Fourier transform yields for all $x \in \mathbb{R}$ and $\lambda \geq \lambda_0$

$$\begin{aligned} \left(v^{(\varepsilon)}(x)\right)^2 &\leq \\ \frac{N_1}{4\pi^2} \int_{\mathbb{R}} |\hat{v}^{(\varepsilon)}(\xi)|^2 \left([K|\xi|^\alpha + \lambda]^2 + |\xi|^2\right)^2 d\xi &\leq \\ \frac{N_1}{\pi^2} \int_{\mathbb{R}} \left(f^{(\varepsilon)}(z)\right)^2 dz. & \end{aligned}$$

The result follows then by taking the limit $\varepsilon \rightarrow 0$ in the above inequality and using the Lebesgue dominated convergence theorem. \square

Now we assume that there exist a constant $K > 0$ such that

$$[a^{-1}|b|^\alpha](x) \leq K \quad \text{for all } x \in \mathbb{R} \quad (4.8)$$

and we are interested in L_2 - estimates of the form

$$\mathbf{E} \int_0^\infty e^{-\lambda s} f(x_0 + L_s) ds \leq N \|f\|_2. \quad (4.9)$$

Theorem 4.2 *Let L be a solution of the equation (1.3) and the assumption (4.8) is true. Then, for any $x_0 \in \mathbb{R}$, $\lambda \geq \lambda_0$, and any measurable function $f : \mathbb{R} \rightarrow [0, \infty)$, the estimate (4.9) holds where the constant N depends on K and α only.*

Proof. Assume first that $f \in C_0^\infty(\mathbb{R})$ so that there is a solution v of equation (4.1) satisfying the inequality (4.3). By taking the ε -convolution on both sides of (4.1), we obtain for all $0 \leq r \leq K$

$$r\mathcal{L}v^{(\varepsilon)} - \lambda v^{(\varepsilon)} + v_x^{(\varepsilon)} + f^{(\varepsilon)} \leq 0.$$

Therefore, for $s \geq 0$, applying the Itô's formula to the expression

$$v^{(\varepsilon)}(x_0 + L_s)e^{-\lambda s},$$

yields

$$\begin{aligned} \mathbf{E}v^{(\varepsilon)}(x_0 + L_s)e^{-\lambda s} - v^{(\varepsilon)}(x_0) &= \\ \mathbf{E} \int_0^s e^{-\lambda u} \left([a^{-1}|b|^\alpha](L_u)\mathcal{L}v^{(\varepsilon)} - \lambda v^{(\varepsilon)} + v_x^{(\varepsilon)}\right)(x_0 + L_u) du &\leq \\ -\mathbf{E} \int_0^s e^{-\lambda u} f^{(\varepsilon)}(x_0 + L_u) du. & \end{aligned}$$

By Lemma 4.1

$$\mathbf{E} \int_0^s e^{-\lambda u} f^{(\varepsilon)}(x_0 + L_u) du \leq \sup_{x_0} v^{(\varepsilon)}(x_0) \leq N \|f^{(\varepsilon)}\|_2.$$

It remains to pass to the limit in the above inequality letting $\varepsilon \rightarrow 0, s \rightarrow \infty$ and using the Fatou's lemma.

The inequality (4.9) can be extended in a standard way first to any function $f \in L_2(\mathbb{R})$ and then to any nonnegative, measurable function using the monotone class theorem arguments (see, for example, [4], Theorem 20). \square

Corollary 4.3 *Let L be a solution of the equation (1.3) and the assumption (4.8) is true. Then, for any $x_0 \in \mathbb{R}, \lambda \geq \lambda_0, m \in \mathbb{N}, t \geq 0$, and any measurable function $f : \mathbb{R} \rightarrow [0, \infty)$, it holds*

$$\mathbf{E} \int_0^{t \wedge \tau_m(L)} f(x_0 + L_s) ds \leq N \|f\|_{2,m} := N \left(\int_{[-m,m]} f^2(y) dy \right)^{1/2},$$

where where $\tau_m(L) = \inf\{t \geq 0 : |x_0 + L_t| > m\}$ and the constant N depends on K, m, t , and α only.

5 Existence of solutions

Here we shall first prove the existence of solutions of equation (1.3) with $0 < \alpha < 1$ using the estimates derived in the previous section. Combined with the results of section 3, it will allow us to formulate the corresponding results for the existence of solutions of equation (1.1).

Theorem 5.1 *Assume that $0 < \alpha < 1$ and the assumption (4.8) is satisfied. Then, for any $x_0 \in \mathbb{R}$, there exists a solution of the equation (1.3).*

Proof. First, by assumption (4.8), there exists a sequence of functions $h_n, n \geq 1$, being Lipschitz continuous and uniformly bounded by K such that $h_n \rightarrow [a^{-1/\alpha}b]$ as $n \rightarrow \infty$ pointwise and in $\|\cdot\|_{2,m}$ -norm for all $m \in \mathbb{N}$. For any $n = 1, 2, \dots$, the equation

$$dL_t^n = h_n(L_{t-}^n) dZ_t + dt \tag{5.1}$$

has a unique strong solution (see, for example, Theorem 9.1 in [9]) where the process Z is defined on a priori fixed probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Our goal is to show that the sequence of processes $\{L^n\}, n \geq 1$ converges to a process L that satisfies the equation (1.3).

Let

$$Y_t^n := \int_0^t h_n(L_{s-}^n) dZ_s.$$

We shall show that the sequence of processes $Q^n := (L^n, Y^n, Z)$, $n \geq 1$, is tight in the sense of weak convergence in $(\mathbf{D}^3, \mathcal{D}^3)$. Due to the Aldous' criterion ([1]), it is enough to verify that

$$\lim_{C \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P} \left(\sup_{0 \leq s \leq t} \|Q_s^n\| > C \right) = 0 \quad (5.2)$$

for all $t \geq 0$ and

$$\limsup_{n \rightarrow \infty} \mathbf{P} \left(\|Q_{t \wedge (\tau^n + r_n)}^n - Q_{t \wedge \tau^n}^n\| > \varepsilon \right) = 0 \quad (5.3)$$

for all $t \geq 0, \varepsilon > 0$, every sequence of \mathbb{F} -stopping times τ^n , and every sequence of real numbers r_n such that $r_n \downarrow 0$. We use $\|\cdot\|$ to denote the Euclidean norm of a vector.

On another hand, for the tightness of the sequence Q^n , it suffices to prove the tightness of the sequence of processes R^n where

$$R_t^n = \int_0^t |h_n|^\alpha (L_s^n) ds.$$

However, the sequence of processes R^n trivially satisfies the Aldous' conditions because of the uniform boundness of the coefficients h_n for all $n \geq 1$.

Since the sequence $\{Q^n\}$ is tight, there exists a subsequence $\{n_k\}, k = 1, 2, \dots$, a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ and the process \tilde{Q} on it with values in $(\mathbf{D}^3, \mathcal{D}^3)$ such that Q^{n_k} converges weakly (in distribution) to the process \tilde{Q} as $k \rightarrow \infty$. For simplicity, let $\{n_k\} = \{n\}$.

According to the embedding principle of Skorokhod (see, e.g. Theorem 2.7 in [9]), there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ and the processes $\tilde{Q} = (\tilde{L}, \tilde{Y}, \tilde{Z})$, $\tilde{Q}^n = (\tilde{L}^n, \tilde{Y}^n, \tilde{Z}^n)$, $n = 1, 2, \dots$, on it such that

- i) $\tilde{Q}^n \rightarrow \tilde{Q}$ as $n \rightarrow \infty$ $\tilde{\mathbf{P}}$ -a.s.
- ii) $\tilde{Q}^n = Q^n$ in distribution for all $n = 1, 2, \dots$

Using standard measurability arguments ([13], chapter 2), one can prove that the processes \tilde{Z}^n, \tilde{Z} are symmetric stable processes of the given index $0 < \alpha < 1$ with respect to the augmented filtrations $\tilde{\mathbb{F}}^n$ and $\tilde{\mathbb{F}}$ generated by processes \tilde{Q}^n and \tilde{Q} , respectively.

Using the properties i), ii), and the equation (5.1), one can show (cf. [13], chapter 2) that

$$\tilde{L}_t^n = x_0 + \int_0^t h_n(\tilde{L}_{s-}^n) d\tilde{Z}_s^n + dt, \quad t \geq 0, \quad \tilde{\mathbf{P}}\text{-a.s.}$$

On the other hand, the same properties and the quasi-left continuity of the the processes \tilde{Q}^n yield

$$\lim_{n \rightarrow \infty} \tilde{L}_t^n = \tilde{L}_t \quad \tilde{\mathbf{P}}\text{-a.s.} \quad (5.4)$$

Therefore, in order to show that the process \tilde{L} is a solution of the equation (1.3), it suffices to verify that, for all $t \geq 0$,

$$\lim_{n \rightarrow \infty} \int_0^t h_n(\tilde{L}_{s-}^n) d\tilde{Z}_s^n = \int_0^t [a^{-1/\alpha} b](\tilde{L}_{s-}) d\tilde{Z}_s \quad \tilde{\mathbf{P}}\text{- a.s.} \quad (5.5)$$

The following fact can be proven similar as Lemma 4.2 in [11].

Lemma 5.2 *For any Borel measurable function $f : \mathbb{R} \rightarrow [0, \infty)$ and any $t \geq 0$, there exists a sequence $m_k \in (0, \infty)$, $k = 1, 2, \dots$ such that $m_k \uparrow \infty$ as $k \rightarrow \infty$ and it holds*

$$\tilde{\mathbf{E}} \int_0^{t \wedge \tau_{m_k}(\tilde{L})} f(\tilde{L}_s) ds \leq N \|f\|_{2, m_k},$$

where the constant N depends on K, α, t , and m_k only.

Without loss of generality, we can assume in Lemma 5.2 that $\{m_k\} = \{m\}$. For (5.5) to be true, it is enough to verify that for all $t \geq 0$ and $\varepsilon > 0$ we have

$$\lim_{n \rightarrow \infty} \tilde{\mathbf{P}} \left(\left| \int_0^t h_n(\tilde{L}_{s-}^n) d\tilde{Z}_s^n - \int_0^t [a^{-1/\alpha} b](\tilde{L}_{s-}) d\tilde{Z}_s \right| > \varepsilon \right) = 0. \quad (5.6)$$

In order to show (5.6) we estimate for a fixed $n_0 \in \mathbb{N}$

$$\begin{aligned} & \tilde{\mathbf{P}} \left(\left| \int_0^t h_n(\tilde{L}_{s-}^n) d\tilde{Z}_s^n - \int_0^t [a^{-1/\alpha} b](\tilde{L}_{s-}) d\tilde{Z}_s \right| > \varepsilon \right) \leq \\ & \tilde{\mathbf{P}} \left(\left| \int_0^t h_{n_0}(\tilde{L}_{s-}^n) d\tilde{Z}_s^n - \int_0^t h_{n_0}(\tilde{L}_{s-}) d\tilde{Z}_s \right| > \frac{\varepsilon}{3} \right) \\ & + \tilde{\mathbf{P}} \left(\left| \int_0^{t \wedge \tau_m(\tilde{L}^n)} [h_n(\tilde{L}_{s-}^n) - h_{n_0}(\tilde{L}_{s-}^n)] d\tilde{Z}_s^n \right| > \frac{\varepsilon}{3} \right) \\ & + \tilde{\mathbf{P}} \left(\left| \int_0^{t \wedge \tau_m(\tilde{L})} [h_{n_0}(\tilde{L}_{s-}) - [a^{-1/\alpha} b](\tilde{L}_{s-})] d\tilde{Z}_s \right| > \frac{\varepsilon}{3} \right) \\ & + \tilde{\mathbf{P}} \left(\tau_m(\tilde{L}^n) < t \right) + \tilde{\mathbf{P}} \left(\tau_m(\tilde{L}) < t \right) = \\ & J_{n, n_0}^1 + J_{n, n_0, m}^2 + J_{n_0, m}^3 + \tilde{\mathbf{P}} \left(\tau_m(\tilde{L}^n) < t \right) + \tilde{\mathbf{P}} \left(\tau_m(\tilde{L}) < t \right). \end{aligned}$$

By fixed n_0 , $J_{n, n_0}^1 \rightarrow 0$ as $n \rightarrow \infty$ by Skorokhod's Lemma about the convergence of stochastic integrals with respect to symmetric stable processes (cf. Lemma 2.3 in [15]). To show that $J_{n, n_0, m}^2 \rightarrow 0$ as $n \rightarrow \infty$ and $J_{n_0, m}^3 \rightarrow 0$ as $n_0 \rightarrow \infty$, we use first the Chebyshev's inequality and (2.1) and then Theorem 4.2 and Lemma 5.2, respectively, to estimate

$$J_{n, n_0, m}^2 \leq \frac{3C_\alpha}{\varepsilon} N \| |h_n - h_{n_0}|^\alpha \|_{2, m} \quad (5.7)$$

and

$$J_{n_0, m}^3 \leq \frac{3C_\alpha}{\varepsilon} N \| |h_{n_0} - [a^{-1/\alpha}b]^\alpha | \|_{2, m} \quad (5.8)$$

where the constant N depends on K, α, m , and t only. Obviously, $\| |h_n - [a^{-1/\alpha}b]^\alpha | \|_{2, m} \rightarrow 0$ as $n \rightarrow \infty$ implying that the right-hand sides in (5.7) and (5.8) converge to 0 by letting first $n \rightarrow \infty$ and then $n_0 \rightarrow \infty$.

Because of the property $\tau_m(\tilde{L}^n) \rightarrow \tau_m(\tilde{L})$ as $n \rightarrow \infty$ $\tilde{\mathbf{P}}$ -a.s.,

$$\tilde{\mathbf{P}}\left(\tau_m(\tilde{L}^n) < t\right) \rightarrow \tilde{\mathbf{P}}\left(\tau_m(\tilde{L}) < t\right) \quad \text{as } n \rightarrow \infty$$

for all $m \in \mathbb{N}, t > 0$. Therefore, the last two terms can be made arbitrarily small by choosing large enough m for all n due to the fact that the sequence of processes \tilde{L}^n satisfies the property (5.2). This proves (5.6).

□

From Theorem 5.1 and Propositions 3.1 and 3.2 we obtain

Theorem 5.3 *Let $0 < \alpha < 1$ and there exist strictly positive constants δ_1, δ_2, K_1 , and K_2 such that*

- 1) $\delta_1 \leq |b|(x) \leq \delta_2$ for all $x \in \mathbb{R}$;
- 2) $K_1 \leq a(x) \leq K_2$ for all $x \in \mathbb{R}$.

Then, for any initial value $x_0 \in \mathbb{R}$, equation (1.1) has a solution.

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