STATEMENT OF RESEARCH INTERESTS

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My current research activities and interests are concentrated in the following areas:

1. Stochastic differential equations with respect to Levy processes
2. Stochastic equations of Doeblin type
3. Weak solutions of SDEs over the field of p-adic numbers
4. Stochastic differential inclusions
5. Applications of stochastic models

1. Stochastic differential equations with respect to Levy processes

One considers the stochastic differential equation

$$dX_t = b(t, X_{t-})dZ_t + a(t, X_t)dt, \quad t \geq 0, \tag{1}$$

where $X_0 = x_0 \in \mathbb{R}^d$ is an initial value, $b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^q, a : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ are measurable functions, Z is a q-dimensional Levy process, and $q \geq 1, d \geq 1$. Since a Levy process is a semimartingale, the coresponding stochastic integral in (1) is well-defined via semimartingale integration theory.

The case of Brownian motion

If Z is a process of Brownian motion then the equation (1) is often refered to as Itó stochastic differential equation. The theory of equations (1) in this case is originated in the works by K. Itó [14], [15] and is considered to be quite accomplished to this time. We also know a lot about the existence and uniqueness of solutions even though one distinguish so-called strong and weak solutions (any strong solution is a weak solution but not vise versa). In the simplest case, the existence/uniqueness problem for weak solutions is solved completely. In other words, if we consider the one-dimensional ($d = q = 1$) time-independent equation (1) without drift ($a = 0$), then the following result is true (see [12]):

Theorem 1. (Engelbert and Schmidt, 1991) *For any initial value $x_0 \in \mathbb{R}$ there exists a (weak) solution of equation (1) if and only if $M \subset N$. Moreover, a solution of the equation (1) is unique in distribution if and only if $N \subset M$,*

where the sets N and M are defined as follows:

$$N := \{x \in \mathbb{R} : b(x) = 0\},$$

$$M := \{x \in \mathbb{R} : \int_{x-\varepsilon}^{x+\varepsilon} b^{-2}(y)dy = \infty \quad \text{for all } \varepsilon > 0\}.$$

The condition $M \subset N$ means for the coefficient b that the local integrability of b^{-2} can get lost just in the points x where $b(x) = 0$. In particular, if we assume that b^{-2} is locally integrable, then $M = \emptyset$ and for any initial value $x_0 \in \mathbb{R}$ there is a non-trivial (not equal to a constant a.s.) solution.

There have been many attemps in trying to generalize the Engelbert-Schmidt conditions to more general cases of equation (1). In particular, in the case of the multidimensional Itó equation (1) without drift, one was able to prove the following result. To formulate it, assume that f is a measurable function on $[0, +\infty) \times \mathbb{R}^d$ and we use the notation $f \in L^{\text{loc}}([0, +\infty) \times \mathbb{R}^d)$ to indicate that f is locally integrable, i.e., integrable with respect to the Lebesgue measure on every compact subset of $[0, +\infty) \times \mathbb{R}^d$. We also define the measure μ on $[0, +\infty) \times \mathbb{R}^d$ by

$$d\mu(s, y) = 1/2[\det bb^*(s, y)]^{-1}\, dy\, ds$$

where $0^{-1} = +\infty$ and b^* is the transpose matrix of b. Similarly, the notation $f \in L^{\text{loc}}([0, +\infty) \times \mathbb{R}^d, \mu)$ stands for the local integrability of f with respect to the measure μ on $[0, +\infty) \times \mathbb{R}^d$.

We need the following three conditions:

- $\mathbf{a_1}$) $(\det bb^*)^{-1} \in L^{\text{loc}}([0, +\infty) \times \mathbb{R}^d)$.
- $\mathbf{a_2}$) $\|b\|^{2(d+1)} \in L^{\text{loc}}([0, +\infty) \times \mathbb{R}^d, \mu)$.
- $\mathbf{b}$) $|a|^{d+1}(\|b\|^{2d} + 1) \in L^{\text{loc}}([0, +\infty) \times \mathbb{R}^d, \mu)$.

Then we have (see [8])

Theorem 2. (Engelbert and Kurenok, 2000) Suppose that the conditions $\mathbf{a_1}$) and $\mathbf{a_2}$) are satisfied. Then, for an arbitrary $x_0 \in \mathbb{R}^d$, there exists a solution X of equation (1) without drift with $X_0 = x_0$.

The result of the Theorem 2 was generalized recently for the case of the equation (1) with drift (cf. [26], [30]:

Theorem 3. (Kurenok and Lepeyev, 2004) Suppose that the conditions $\mathbf{a_1}), \mathbf{a_2}$) and $\mathbf{b}$) are satisfied. Then, for an arbitrary $x_0 \in \mathbb{R}^d$, there exists a solution X of the equation (1) with $X_0 = x_0$.

It should be pointed out that both theorems, Theorem 2 and Theorem 3, garantee the existence of solutions that can, in general, explode, that is, they can leave any compact set on a finite time interval. It is, in particular, a consequence of integrability conditions that have local character. On another side, there are known the results of N. I. Portenko (see, for example [42]) who constructed global solutions of equation (1) with drift satisfying some global integrability conditions, that is $a \in L_p(\mathbb{R}^{d+1})$ for some suitable p.

The proof techniques used by Engelbert/Kurenok and Kurenok/Lepeyev are different from those used by Portenko. Therefore, in both cases, for the equation (1) without drift and with drift, it would be interesting

• *to find additional conditions that ensure along with the conditions* a_1) *and* a_2) *(or conditions* a_1), a_2), *and* b), *respectively) the existence of a global solution.*

Additionally, it is interesting

• *to classify the set of all solutions in multidimensional case in terms of the uniqueness in law and the occurance of explosion. In other words, to find in the set of all solutions those that do have the same distribution and those that do not explode.*

It is worth to do even in the time-independent case of the equation (1) without drift. Such description is already known for the one-dimensional time-independent equation (1) without drift (see [11]).

Finally, the techniques we used to investigate the weak solutions of (1), can be applied to the concept of strong solutions of the equation (1). For example, for the case of the equation (1) with the unit diffusion coefficient,

$$dX_t = dW_t + a(t, X_t)dt, \quad X_0 \in \mathbb{R}^{d+1},$$

N. V. Krylov and M. Röckner [22] were able to prove the existence of a strong solution just under some local integrability condition of the coefficient a.

I am planning

• *to apply the technique of Krylov's estimates used in the case of weak solutions to the concept of strong solutions for the general equation (1) to possibly obtain the results generalizing the results of Krylov and Röckner.*

The case of symmetric stable processes of index $0 < \alpha \leq 2$

A (one-dimensional) symmetric stable process of index $\alpha \in (0, 2]$ is a Levy process Z such that

$$\mathbf{E}e^{i\lambda Z_t} = e^{-t|\lambda|^\alpha}$$

for all $t \geq 0, \lambda \in \mathbb{R}$.

If $\alpha = 2$ then it is a Brownian motion with variance $2t$ and it is the only symmetric stable process with continuous trajectories. For all other values of α the process Z is a purely discontinuous semimartingale; in particular, for $\alpha = 1$ it is a well-known Cauchy process. There is a lot less known about stochastic differential equations of form (1) driven by a symmetric stable process Z with $0 < \alpha < 2$ rather than with $\alpha = 2$. Partially, it is due to the fact that the process Z has in the case of $0 < \alpha < 2$ "not very good analytical properties" as in the case $\alpha = 2$. From another side, the interest to equations (1) with $0 < \alpha < 2$ has risen substantially in recent years because of applications of such kind of models: in the risk theory, financial mathematics and other areas where the stochastic models of jump-type seem to be more adequate than continuous ones.

The one-dimensional time-independent equation

$$dX_t = b(X_{t-})dZ_t, \quad X_0 = x_0 \in \mathbb{R} \tag{2}$$

was studied in detail by P. A. Zanzotto [50], [51] who was able to generalize completely the results of Engelbert and Schmidt (Theorem 1) for the case $1 < \alpha \le 2$:

Theorem 4. (Zanzotto, 2002) Let $1 < \alpha \le 2$. Then, for any initial value $x_0 \in \mathbb{R}$, there exists a unique in law (weak) solution of the equation (2) if and only if $M_\alpha = N$ where the set M_α is defined as

$$M_\alpha := \{x \in \mathbb{R} : \int_{x-\varepsilon}^{x+\varepsilon} |b|^{-\alpha}(y)dy = \infty \quad \text{for all } \varepsilon > 0\}.$$

The case of time-dependent equation

$$dX_t = b(t, X_{t-})dZ_t, \quad X_0 = x_0 \in \mathbb{R} \tag{3}$$

is more complex and there are found to the present moment some different sufficient conditions ensuring the existence of solutions. First, H. Pragarauskas and P. A. Zanzotto [46] considered the case of the equation (3) with $1 < \alpha < 2$ where they used the method of Krylov's estimates for stable integrals to construct a solution. Their main result is

Theorem 5. (Pragarauskas and Zanzotto, 2000) Let $1 < \alpha \le 2$ and assume that b satisfies the following three conditions:

(i) for any $t > 0$ and $m > 0$,

$$\int_0^t \int_{-m}^m |b(s, y)|^\alpha ds dy < \infty,$$

(ii) for any $t > 0$ and $m > 0$,

$$\int_0^t \int_{-m}^m |b(s, y)|^{-\alpha} ds dy < \infty,$$

(iii) for any $t > 0$

$$\lambda\{y : \sup_{0 \le s \le t} |b(s, y)| < \infty\} > 0,$$

where λ denotes the Lebesgue measure on $\mathbb{R}$.

Then, for any $x_0 \in \mathbb{R}$, there exists a non-exploding solution of the equation (3).

We notice that the condition (iii) is needed to garantee that the constructed solution will not explode. The conditions (i) and (ii) ensure the existence of a (possibly, exploding) solution.

On another hand, H. J. Engelbert and V. P. Kurenok [9] studied the general case of $0 < \alpha < 2$ where they used the time change method to construct a solution. Both exploding and non-exploding solutions were treated. We refer to [9] for detailed formulation of results.

Recently [31] I was also able to generalize the result of Pragarauskas and Zanzotto in the following form:

Theorem 6. (Kurenok, 2007) Let $1 < \alpha \leq 2$. Suppose that the following two conditions are satisfied:

(i) for any $t > 0$ and $m > 0$,

$$\int_0^t \int_{-m}^{m} |b(s,y)|^\alpha ds dy < \infty,$$

(ii) $M_\alpha \subset N$,

(iii) for any $t > 0$

$$\lambda\{y : \sup_{0 \leq s \leq t} |b(s,y)| < \infty\} > 0.$$

Then, for any initial value $x_0 \in \mathbb{R}$, there exists a nonexploding solution of the equation (3).

We notice that the sets M_α and N are defined here analogeously as in the time-independent case.

As for stochastic differential equations (2) and (3) with drift and $\alpha \in (0,2)$, there is essential difference how to handle this case to compare with the case of $\alpha = 2$. Because of lack of suitable drift transformation that transforms a symmetric stable process with respect to a given probability measure to another such process with respect to a different probability measure, we cannot handle the equations (2) and (3) with drift using the corresponding equations without drift and a Girsanov transform. However, the solution of the problem was found with help of Krylov's estimates.

It is well-known that the estimates for distrubutions of stochastic integrals are very important in the theory of SDE's and their applications. First estimates of such kind were derived by N. V. Krylov [23] for the processes of diffusion type. Therefore, sometimes they are called Krylov's estimates.

The formulation is the following. Let $f : [0,\infty) \times \mathbb{R} \to [0,\infty]$ be a measurable function. For arbitrary but fixed $t > 0$ and $m \in \mathbb{N}$ we define $\|f\|_{L_p,m,t} = (\int_0^t \int_{[-m,m]} |f(s,x)|^p ds dx)^{\frac{1}{p}}$ being the L_p-norm of f on $[0,t] \times [-m,m], p \geq 1$. Suppose that X is a solution of the equation (1) and $\tau_m(X) = \inf\{t \geq 0 : |X_t| > m\}$.

We are interested in the local estimates of the form

$$\mathbf{E} \int_0^{t \wedge \tau_m(X)} |b|^\alpha (s, X_s) f(s, X_s) ds \leq N \|f\|_{L_p,m,t}, \tag{4}$$

where N is a constant depending on α, m, p, and t only. The global Krylov's estimates of the form (4) are defined correspondingly.

The proof of Krylov's estimates relies on important analytical facts of independent interest. For example, in the time-independent case we procee as follows. For $K > 0$, let

$$dX_t^\gamma = dZ_t + \gamma_t dt$$

where γ is a predictable process bounded by the constant K, and for all $\lambda > 0, x \in \mathbb{R}$, and a Borel measurable function $f : \mathbb{R} \to [0, \infty)$, define the value function

$$v(x) = \sup_{|\gamma| \le K} \mathbf{E} \int_0^\infty e^{-\lambda t} f(x + X_t^\gamma) dt.$$

Let Z be a symmetric stable process of index $1 < \alpha < 2$. It is shown in [27] that if $f \in C_0^\infty$ then the function v satisfies the equation

$$\mathcal{L}v - \lambda v + K|v_x| + f = 0 \tag{5}$$

and

$$\sup_{x \in \mathbb{R}} v(x) \le N\Big(\int_{\mathbb{R}} f^2(y)dy\Big)^{1/2}. \tag{6}$$

Here $\mathcal{L}$ is the infinitesimal generator of the process Z. Moreover, it is shown in [27] that equations

$$dX_t = dZ_t + [a|b|^{-\alpha}](X_t)dt \tag{7}$$

and

$$dX_t = b(X_{t-})dZ_t + a(X_t)dt \tag{8}$$

have nonexploding solutions if

$$|a(x)| \le K|b|^\alpha(x) \quad \text{ for all } x \in \mathbb{R}$$

and the coefficient b is bounded and nondegenerate. On another hand, the solutions of equations (7) and (8) are related to each other by a suitable time change; that is, a solution of one equation can be obtained from a solution of another equation by a corresponding time change.

We also point out here to generalizations of results in [27] for some other models. Thus, in [24] we considered equation (8) with respect to a Cauchy process ($\alpha = 1$). The general time-dependent case of equation (1) but with $\alpha \in (1, 2)$ was treated in [32]. The case with $\alpha < 1$ seems to be a more harder one. There are at present only some partial results for this case - we refer here to [33] and [35].

Finally, using approach described above we also investigated in [34] the following diffusion model with jumjps

$$dX_t = b_1(X_t)dW_t + b_2(X_{t-})dZ_t + a(X_t)dt$$

where $b_1, b_2, a : \mathbb{R} \to \mathbb{R}$ are Borel measurable functions, W is a Brownian motion, and Z is a symmetric stable process of index $\alpha \in (0, 2)$.

It would interesting

- *to obtain Krylov's estimates for solutions X of equation (1) with coefficients being only* **locally bounded** *or possibly satisfying some assumptions of* **integrability/local integrability**
- *to extend the existence and uniqueness results for solutions of equation (1) to SDEs with coefficients satisfying weaker conditions*

Stochastic equations with reflections

Recently (cf.[10]) we began studying the reflected solutions of SDE's driven by symmetric stable processes with $\alpha < 2$. Such models are very important in applications, in particular, in finance where a price process cannot become negative rather it should be reflected into positive half-axis as soon as it reaches the zero boundary.

We considered the model described by the process X of the form

$$X_t = x_0 + \int_0^t b(X_{s-})dZ_s + K_t, \quad t \geq 0 \tag{9}$$

where $X_t \geq 0$ for all $t \geq 0$ and K is a right-continuous, increasing process with $K_0 = 0$ satisfying

$$\int_0^\infty \mathbf{1}_{\{X_s \neq 0\}} dK_s = 0. \tag{10}$$

The relation (10) means that the process K plays the role of a reflecting force for the process X because it increases only when X reaches zero boundary.

To construct a reflected solution of the equation (9), we first constructed a reflected symmetric stable process $\bar{Z}$ as a solution of the equation

$$\bar{Z}_t = \bar{Z}_0 + Z_t + K_t, \quad t \geq 0,$$

where $\bar{Z}_t \geq 0$ for all $t \geq 0$, Z is a symmetric stable process with $Z_0 = 0$, and K is the reflecting force for $\bar{Z}$. It turned out that, for all values of $0 < \alpha \leq 2$, the process $\bar{Z}$ is recurrent. We constructed a solution X of the equation (9) by making a suitable time change in the process $\bar{Z}$ and classified solutions in terms of the uniqueness in distribution.

Some open questions are still left.

- *I am interested in investigating whether the reflecting force K of the solution X can be described in terms of the local time of X.*

We know that the process X is a time changed recurrent process and we can expect that X possesses the local time. As for the case of $\alpha = 2$, it is already known [43] that the reflecting force can be expressed as the local time of the reflected Brownian motion X.

Additionally,

- *I am willing to consider the case of reflected SDE's with two boundaries.*

The case of an arbitrary Levy process

The equation (1) with respect to an arbitrary Levy process Z and *only measurable* coefficients b and a (instead of continuous or Lipshitz) has not been studied so far. One particular case, namely the one-dimensional equation of the form

$$dX_t = dZ_t + a(X_t)dt \tag{11}$$

was considered in [48] where one assumed the coefficient a to be (in general) continuous and to satisfy additionally some assumptions of boundedness (for $\alpha \geq 1$) or to be even smooth (for $\alpha < 1$). The method used in [48] was a purely analytical one based on the analysis of properties of Markov charcteristics of the process X satisfying equation (11).

Our goal remains the same: to study the equation (1) in case of an arbitrary Levy process Z using the approach based of Krylov's estimates. In papers [28] and [29] we were able to derive corresponding Krylov's estimates and prove the existence of solutions for multidimensional equation

$$dX_t = dZ_t + a(t, X_t)dt, \quad X_0 = x_0 \in \mathbb{R}^d, \quad d \geq 1 \tag{12}$$

and a symmetric stable process Z of index $\alpha \in (1, 2)$ under assumption that a is bounded.

- *Generalizing/extending the results from [28] and [29] to equations (1) with a general coefficient b is an open promlem - see also next section.*

2. Stochastic equations of Doeblin type

Let $\beta : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d_+, a : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ be Borel measurable functions and $(Z^1, Z^2, \ldots Z^d)$ be a sequence of independent processes where $d \geq 1$ is the dimension. We denote by β_i and a_i the i-th term of vector-functions β and a, respectively. Consider the equation

$$X^i(t) = Z^i[\int_0^t \beta_i(s, X(s))ds] + \int_0^t a_i(s, X(s))ds, \quad i = 1, 2, \ldots, d \tag{13}$$

where we write for convenience $X(t)$ instead of usual X_t. If $d = 1$ we have

$$X(t) = Z[\int_0^t \beta(s, X(s))ds] + \int_0^t a(s, X(s))ds \tag{14}$$

where β, a, and Z are corresponding one-dimensional objects.

For simplicity, let us look first at the one-dimensional equation. Define

$$\tau(t) := \int_0^t \beta(s, X(s))ds$$

where $(X, \mathbb{F})$ is a given process. One of simple possibilities to give a suitable sense to equation (14) is to require the process τ to be a time change with respect to the filtration $\mathbb{F}$.

We say that equation (14) has a (weak) solution if there is a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a filtration $\mathbb{F}$ and processes $(X, \mathbb{F})$ and $(Z, \mathbb{F})$ on it such that the process τ is a well-defined $\mathbb{F}$-time change and (14) holds $\mathbf{P}$-a.s.

The equation (14) could be called perhaps "random clock" or "time change" equation because of its nature but we call it equation of "Doeblin type". It is now known ([3], [7]) that Wolfgang Doeblin was probably the first who considered equations of that type when Z is a Brownian motion, well before the notions of a stochastic integral and stochastic differential equations were introduced by K. Ito. Equation (14) was used by Doeblin to describe one-dimensional diffusions.

Is there any relation between Doeblin equation (14) and Ito stochastic equation (1)? To answer this question, we need to impose some restrictions on the process Z. First of all, the stochastic integral in (1) is defined only for appropriate classes of processes Z. From this side the assumption on Z being a semimartingale or at least a Levy process seems to be natural. As already mentioned in section 1, symmetric stable processes form an important subclass of Levy processes. It turns out that equations (1) and (14) are roughly speaking equivalent if Z is a symmetric stable process and $\beta = |b|^\alpha$. It follows essentially from the fact that a stochastic integral with respect to a symmetric stable process is nothing but another symmetric stable process of the same index α with a corresponding "inner clock" completely determined by the integrand cf. [47], [9]. From another side, as shown in [16], symmetric stable processes are only among Levy processes who have this property. With other words, equations (1) and (14) are not equivalent if Z is not a symmetric stable process. The mentioned property follows actually from the following well-known fact: for any $b > 0$ and $t > 0$ the distributions of processes $Z(t)$ and $Z(b^\alpha t)/b$ coincide.

In our opinion, Doeblin equation (14) can be seen as an *alternative model* to the stochastic Ito equation when Z is a general Levy process or a semimartingale. The noise part in equation (14) is obtained from putting into driving process Z a corresponding "random clock" called in the financial language the "operational time", the case which is often seen in praxis and is desired to be modeled in many applications.

One nice property of eqution (14) consists in the following. Let X be a solution of (14) and Z is a Levy process with infinitesimal generator $\mathcal{L}$. It follows then that under some mild assumptions on coefficients a and b we have that the infinitesimal generator of X is $\beta\mathcal{L} + a\frac{d}{dx}$ - cf. [36].

Some of future projects concerning the Doeblin equations are the following:

- *Let $d = 1, a = 0$ and Z is a Levy process for that 0 is a regular point. It is then well-known (cf. [2], chapter 5) that Z has the local time process $L^Z(t, x), t \geq 0, x \in \mathbb{R}$. We assume additionally that Z satisfies the assumption that garantees the joint continuity of L^Z in (t, x) (cf. [2], p. 148). The goal is to investigate the behaviour of the following* *integral functionls*

$$T_t = \int\limits_0^t f(Z_s)ds \tag{15}$$

where $f : \mathbb{R} \to [0, \infty]$ *is a measurable function. More precisely, one should attempt to prove the "zero-one" law for the convergence of* $T_t, t \geq 0$ *similar to the "zero-one" law of Engelbert-Schmidt for the case when* Z *is a Brownian motion.*

- *Using the results about the convergence of integral functionals in (15), to investigate the existence and uniqueness of solutions of the equation*

$$X(t) = Z[\int_0^t \beta(s, X(s))ds] \tag{16}$$

where Z *is a Levy process as above.*

- *To investigate more general one-dimensional equation with drift (14) and the multidimensional equation (13).*

3. Weak solutions of SDEs over the field of p-adic numbers

The classical theory of stochastic analysis has been generalized in various directions. Recently, the theory of stochastic analysis over the field of so-called p-adic numbers has become important. It is motivated, in particular, by the needs of mathematical physics [49]. The notion of stochastic integral has been just developed [20], along with first results for stochastic differential equations obtained [17]. In [18] one proved the existence of weak solutions for one-dimensional stochastic equation

$$dX_t = b(X_{t-})dZ_t \tag{17}$$

where Z is a symmetric stable process of index $\alpha \in (1, 2)$ over the field $\mathbf{Q_p}$ of p-adic numbers where the coeficient b is assumed to be locally bounded.

Our goal is

- *To prove the existence of weak solutions of the equation*

$$dX_t = b(t, X_{t-})dZ_t + a(t, X_t)dt \tag{18}$$

where Z *is a one-dimensional symmetric stable process over the field of* p*-adic numbers* $\mathbf{Q_p}$ *and the coefficients* a *and* b *are only measurable. We are intending to use the approach based on Krylov's estimates similar to the case of ordinary SDEs.*

- *To make possible extensions of the results above to the multidimensional case of equation (18).*

4. Stochastic Differential Inclusions

The theory of stochastic differential inclusions is a quite new branch of stochastic analysis and is originated from the theory of differential inclusions for ordinary differential equations. Over the last few decades mathematicians have discovered the possibility to extend the class of equations which have solutions via generalization of the equations to inclusions. This generalization extends the concept of solutions for differential equations to the concept of solutions of corresponding differential inclusions assuming the preservation of the definition in the class of equations which have solutions in the sense of original definition. At the beginning, inclusions were investigated in a close relationship with the corresponding equations but then the theory of inclusions separated from the equations on a new level of abstraction and currently has become a self-contained part of mathematics.

A. F. Filippov [13] was the first who systemized the theory of ordinary differential inclusions. He also pointed out conditions those the inclusions should correspond to.

Stochastic differential inclusions were introduced first by P. Kree [21], though the inclusions had been delt with by E. D. Conway [5]. Since then the stochastic differential inclusions were investigated by several authors. We mention here the contributions of E. Cepa [4], R. Petterson [40], [41], M. Kisielevich [19], A. A. Levakov [38], [39], N. U. Ahmed [1], G. Da Prato and H. Frankowska [6], and A. N. Lepeyev [37].

Let us concentrate on the one-dimensional stochastic differential inclusion of the form

$$dX_t \in B(X_{t-})dZ_t, \quad X_0 = x_0 \in \mathbb{R}, \quad t \geq 0, \tag{19}$$

where $B : \mathbb{R} \to Cl(\mathbb{R})$ is a multivalued Borel measurable mapping, $Cl(\mathbb{R})$ is the set of all closed subsets of $\mathbb{R}$, and Z is a one-dimensional symmetric stable process of index $0 < \alpha \leq 2$.

The stochastic differential inclusion (19) is usually considered in the connection with the corresponding SDE where the righthand side of (19), the multivalued mapping B, is to be constructed in terms of the coefficient b.

A stochastic process X defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is called then a (weak) solution of the inclusion (19) if there exist a symmetric stable process Z of index α and a measurable mapping $u : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that $u(t,\omega) \in B(X_t(\omega)) : \mathbb{R}_+ \times \Omega \to Cl(\mathbb{R}), \quad u(0,\omega) \in B(X_0(\omega))$, and it holds

$$X_t = x_0 + \int_0^t u(s)dZ_s, \quad t \geq 0, \quad \mathbf{P} - \text{a.s.}$$

In all papers mentioned above the stochastic inclusions just for the case of Brownian motion ($\alpha = 2$) were considered, one-dimensional or multidimensional. Additionally, one required there the diffusion coefficient b to be bounded or at least of linear growth. It turns out that, at least in one-dimensional case, we can do more. My former student A. N. Lepeyev [37] proved that in the case of $\alpha = 2$ the stochastic inclusion (19) can be solved for any measurable coefficient b.

- *We are planning to investigate the stochastic differential inclusions (19) of more general forms (time-dependent case, case with drift) and with* $0 < \alpha \leq 2$.

5. Applications of stochastic models

Stochastic dynamical models described by solutions of appropriate stochastic differential equations arise naturally in many applications. One simple example can be taken from modeling of interest rate processes in financial markets.

The interest rate process describes the profitability of some financial instrument, such as stock, bond or option. Hence, if the price change is given by the sequence $X = (X_n)_{n \geq 0}$, then, in the simplest case, the interest rate process has the form

$$r_n = \frac{\triangle X_n}{X_{n-1}}, \text{ where } \triangle X_n = X_n - X_{n-1}.$$

The role of interest rate can better be understood by writing the sequence (X_n) in the form $X_n = X_0 \exp(H_n)$ with $H_n = \sum_{i=1}^n h_i$. It is easily verified that $X_n = X_0 \prod_{i=1}^n (1+r_i)$, where $r_i = \ln(1 + h_i)$. In other words, if X_0 denotes the initial value of the financial sequence X, then the nth value is equal to the product of X_0 and n return-values. Generalizing the above relation for continuous time, we obtain

$$dX_t = r_t X_t dt, \ t \geq 0.$$

In practice, another form of dependence of the processes (r_t) and (X_t) is often used, namely the equation

$$dr_t = \sigma(t, r_t) dX_t, \ t \geq 0,$$

or, more generaly,

$$dr_t = \sigma(t, r_t) dX_t + \gamma(t, r_t) dt, \ t \geq 0. \tag{20}$$

The process (r_t) is called a short interest rate process.

In [24] we proposed the following model for describing the interest rate processes

$$r_t = \mathcal{G}\Big(t, M_{T(t)}\Big), \ t \geq 0, \tag{21}$$

where the process M is a symmetric stable process of the index α, $T(t)$ with $T(0) = 0$ is a continuous strictly increasing function and $\mathcal{G}(t, x)$ is a continuous in both variables function. The model (21) includes, in particular, many interest rate models of the form (20) where X is a symmetric stable process of index $0 < \alpha \leq 2$. Thus, the generalized Hull-White, Black-Karasinski, and Dothan models are particular cases of the model (21).

For the purpose of financial calculations, it is important to know the distributions of the interest rate processes. For example, the contingent claims $C(r_t)_{0 \leq t \leq \tau}$ on the bond markets, depending on (r_t), are usually expressed in the form

$$C(r_t) = \mathbf{E_Q}\Big(\exp\{-\int_t^\tau r_u du\} C(r_\tau) \mid \mathcal{F}_t\Big),$$

where τ is the bond maturity time, $(\mathcal{F}_t)$ is a filtration and Q is an appropriate martingale measure (see, for example, [45], [44]). To calculate $C(r_t)$ we need to know the distribution of the process (r_t). For the pricing of the contingent claims, it would be usefull to have a sequence of more simple processes $(r_t^n),\ n \geq 1$, which converges in a suitable sense to the process (r_t). It should allow us to use the corresponding sequence

$$\mathbf{E_Q}\Big(\exp\{-\int_t^\tau r_u^n du\} C(r_\tau^n) \mid \mathcal{F}_t\Big),\ n \geq 1,$$

for approximating $C(r_t)$.

In [24] we constructed a sequence of processes (r_t^n) that approximates the process (r_t). More precesely, for every fixed n the process (r_t^n) is determined as a sum consisting of only n independent identically distributed random variables and it holds $r^n \to^{\mathcal{D}} r$ for $n \to \infty$ which means the convergence of distributions of the processes r^n to the distribution of the process r.

- *Currenntly I am looking for financial data that could fit to a particular case of the general proposed model (21). After that the calculation formulas derived in [24] could be used to obtain other important characteristics of the model the data is coming from.*

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