
On Existence and Uniqueness of Reflected Solutions of Stochastic Equations Driven by Symmetric Stable Processes *

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Dedicated to Albert Shiryaev in occasion of his 70th birthday

We study the one-dimensional stochastic differential equation (SDE) of the form $X_t = x_0 + \int_0^t b(X_{s-}) dM_s + K_t$, $t \geq 0$, where the volatility $b : [0, \infty) \rightarrow \mathbb{R}$ is a Borel measurable function, $x_0 \in [0, \infty)$ is an arbitrary initial value, the process X is nonnegative, K is a right-continuous increasing process with $K_0 = 0$, and M is a symmetric stable process of arbitrary stability index $0 < \alpha \leq 2$ with $M_0 = 0$. The process K satisfies the condition $\int_0^\infty \mathbf{1}_{\{X_t \neq 0\}} dK_t = 0$, that means that K is the reflecting force for the solution X . For every $x_0 \in [0, \infty)$ we state conditions on b for the existence and uniqueness of a reflected solution X with $X_0 = x_0$. In particular, our results generalize the results of W. M. Schmidt [16] who considered the given SDE in the case of the Brownian motion ($\alpha = 2$).

Key words: Symmetric stable processes, Skorohod reflection problem, integral functionals, stochastic stable integrals, stochastic equations, existence and uniqueness of solutions

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1 Introduction

In this paper we consider the one-dimensional stochastic equation

$$X_t = x_0 + \int_0^t b(X_{s-}) dM_s + K_t, \quad t \geq 0, \quad (1)$$

where the volatility $b : [0, \infty) \rightarrow \mathbb{R}$ is a Borel measurable function, $x_0 \in [0, \infty)$ is an arbitrary initial value, the process X is nonnegative, K (called the reflecting force) is a right-continuous increasing process with $K_0 = 0$ and such that $\int_0^\infty \mathbf{1}_{\{X_t \neq 0\}} dK_t = 0$, and M is a symmetric stable process with $M_0 = 0$.

It is well-known that every symmetric stable process can uniquely be characterized by its stability index $\alpha \in (0, 2]$ in the following sense. A process M is a *symmetric stable process* of index α iff it is a process with homogeneous and independent increments and the characteristic function of M_t has the form

$$\mathbf{E} \exp(i\lambda M_t) = \exp(-t |\lambda|^\alpha), \quad \lambda \in \mathbb{R}, \quad t \geq 0. \quad (2)$$

For $\alpha = 2$ the process M is a Brownian motion (with variance function $2t$) and for $\alpha = 1$ it is a Cauchy process. The Brownian motion is the only symmetric stable process with continuous sample paths. For all other parameters $\alpha \in (0, 2)$ the process M is a *purely discontinuous* semimartingale with infinite variance. Therefore, the cases $\alpha = 2$ and $0 < \alpha < 2$ are rather different, at least from this point of view. The density function for symmetric stable processes can be written in explicit form only in three cases: for the Brownian motion, the Cauchy process and the $\frac{1}{2}$ -stable process. For more details about symmetric stable processes we refer to the well-known books [1], [15] or [18].

By a *stochastic basis* we understand a complete probability space (Ω, \mathcal{F}, P) with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying *the usual conditions*. Now, a *symmetric α -stable process with respect to the filtration \mathbb{F}* is a symmetric α -stable process M which is \mathbb{F} -adapted and such that $M_t - M_s$ is independent of \mathcal{F}_s for all $0 \leq s \leq t$. (Alternatively, $\exp(i\lambda M_t + t |\lambda|^\alpha)$ is a complex valued \mathbb{F} -martingale for every $\lambda \in \mathbb{R}$.) For the sake of simplicity, in that case we say that (M, \mathbb{F}) is a symmetric α -stable process.

Let $\alpha \in (0, 2]$, $x_0 \in [0, \infty)$, and a Borel measurable function $b : [0, \infty) \rightarrow \mathbb{R}$ be fixed.

Definition 1. A process X defined on a stochastic basis $(\Omega, \mathcal{F}, P; \mathbb{F})$ is said to be a *reflected solution of SDE (1) in $[0, \infty)$* if there exist two processes M and K such that:

- 1) X is \mathbb{F} -adapted and $X_t \geq 0$ for all $t \geq 0$;
- 2) (M, \mathbb{F}) is a symmetric stable process of index α ;
- 3) K is an \mathbb{F} -adapted, right-continuous and increasing process with $K_0 = 0$;
- 4) $\int_0^\infty \mathbf{1}_{\{X_t \neq 0\}} dK_t = 0$;
- 5) equation (1) is satisfied P -a.s.

Relation 4) means that K increases only if X becomes zero. The process K is called the *reflecting force* for the solution X of SDE (1).

Under the integral in (1) we may understand the stochastic integral with respect to the symmetric stable process M in the sense of Itô as defined by J. Rosiński and W. Wołczyński [13]. There is a great analogy between the construction of this stochastic integral and the Itô integral for the Brownian motion. However, and this is very important, the result is completely the same if this integral is constructed as a stochastic integral with respect to the semimartingale M , as in the book of J. Jacod and A.N. Shiryaev [7]. In both [7] (Chapter III, 6d) and [13] it was proven that the finiteness of $\int_0^t |b(X_{s-})|^\alpha ds$, for all $t \geq 0$, is necessary and sufficient for the existence of the stochastic integral in (1).

Multidimensional stochastic differential equations with reflections in general form driven by Brownian motion were considered by many authors. We only refer to the papers of L. Słomiński [19], [20], and A. Rozkosz and L. Słomiński [14] where they investigated the equation under quite general assumptions on the coefficients and where one can find other references on this topic. In the one-dimensional case one can obtain more. Equation (1) with driving process M being a Brownian motion was studied in detail by W. M. Schmidt in [16], where he obtained necessary and sufficient conditions for the existence of solutions. W. M. Schmidt essentially used the time change method and the properties of the local time of the Brownian motion. Nonreflected SDEs driven by symmetric α -stable processes were considered by P. A. Zanzotto [23], [24] (time-independent case), by H. Pragarauskas and P. A. Zanzotto [10] (time-dependent case, $1 < \alpha < 2$) and by H. J. Engelbert and V. P. Kurenok [5] (time-dependent case, $0 < \alpha \leq 2$).

The aim of the present paper is to solve SDE (1) for an arbitrary stability index $\alpha \in (0, 2]$. Our results about the existence of solutions will generalize, in particular, the results of W. M. Schmidt for the case $\alpha = 2$. To construct a solution of (1) we use the time change method analogously to the case of nonreflected SDEs (see, e.g., [5]).

The paper is organized as follows. In Section 2 we construct a symmetric stable process with a reflecting boundary at zero for an arbitrary parameter $\alpha \in (0, 2]$. The method for the construction of a Brownian motion with reflecting boundaries used by W. M. Schmidt cannot be applied to the case $0 < \alpha < 2$ because for $0 < \alpha \leq 1$ there doesn't exist a local time process and for $1 < \alpha < 2$ (when the local time exists) there is not (at least in an explicit form) a Tanaka formula as for the case of the Brownian motion. For the general case we use the approach for the construction of a reflected process in a bounded region given by A. V. Skorohod [17]. We prove that the reflected α -stable processes are recurrent for every $\alpha \in (0, 2]$. This allows us to construct nonexploding solutions of (1); that is a situation different from the nonreflected case (when $0 < \alpha < 1$). We also investigate some properties of integral functionals of reflected symmetric stable processes which are the

key for the construction of a solution of (1). They are collected in Section 3. The last section is devoted to the existence and uniqueness of solutions of (1).

2 Reflected symmetric stable processes

In this section we shall construct (in the sense of trajectories) a symmetric stable process reflected at the boundary zero to the right. First of all, we define what we understand by such a process.

Definition 2. *A process \bar{M} with $\bar{M}_0 \geq 0$ is called a reflected symmetric stable process of index α on $[0, \infty)$ if there exist processes M and K such that:*

- 1) M is a symmetric stable process of index α ;
- 2) $\bar{M}_t \geq 0$ for all $t \geq 0$;
- 3) K is an increasing, right-continuous process with $K_0 = 0$ and

$$\int_0^\infty \mathbf{1}_{\{\bar{M}_t \neq 0\}} dK_t = 0;$$

- 4) it holds

$$\bar{M}_t = \bar{M}_0 + M_t + K_t, \quad t \geq 0. \quad (3)$$

As in the previous section, the process K is called the *reflecting force* for \bar{M} .

In the case $\alpha = 2$, the reflected process can be described by $|M|$, where M is a Brownian motion (with variance function $2t$). Using the Tanaka formula, we recover the reflecting force K as the local time of M (or $\frac{1}{2}$ of the local time of $|M|$). This well-known fact was exploited by W. M. Schmidt [16].

But there is another possibility to obtain the reflected Brownian motion which immediately follows from the solution of *the deterministic Skorohod problem*. Let \mathbb{D} be the space of functions $x : [0, \infty) \rightarrow \mathbb{R}$ which are càdlàg (right-continuous, with finite left-hand limits). Then the deterministic Skorohod problem can be formulated as follows. For a given function $x \in \mathbb{D}$ such that $x(0) \geq 0$ there are to find functions z and y from the space \mathbb{D} such that:

- 1) $z(t) \geq 0$ for all $t \geq 0$;
- 2) y is an increasing function with $y(0) = 0$ and $\int_0^\infty \mathbf{1}_{\{z(t) \neq 0\}} dy(t) = 0$;
- 3) it holds $z(t) = x(t) + y(t)$ for all $t \geq 0$.

The existence and uniqueness of the solution of this problem for the space of continuous functions was first proven by A. V. Skorohod [17] in 1961. H. Tanaka [21] generalized the problem by formulating it in the space of càdlàg functions, in the multi-dimensional case. In this generality he proved only uniqueness (Lemma 2.3, [21]). However, in the one-dimensional case, existence also holds and we give a proof, for the convenience of the reader.

Lemma 1. *For every function $x \in \mathbb{D}$ such that $x(0) \geq 0$, the deterministic Skorohod problem on $[0, \infty)$ has a unique solution (z, y) , given by*

$$y(t) := \sup_{0 \leq s \leq t} \max(-x(s), 0), \quad z(t) := x(t) + y(t), \quad t \geq 0. \quad (4)$$

Proof. We first prove existence. Let $x \in \mathbb{D}$ such that $x(0) \geq 0$ and y, z be defined by (4). It is obvious that y, z are in \mathbb{D} , $z(t) \geq 0$ for all $t \geq 0$ and y is an increasing function with $y(0) = 0$. It remains only to prove that $\int_0^\infty \mathbf{1}_{\{z(t) > 0\}} dy(t) = 0$.

Let $\varepsilon > 0$ be fixed, but arbitrary. Since $z \in \mathbb{D}$, the set $\{t \geq 0 : z(t) > \varepsilon\}$ can be written as $\bigcup_{n \geq 1} I_n$, where $I_n, n \geq 1$, are pairwise disjoint intervals of the form $I_n = (u_n, v_n)$ or $[u_n, v_n)$.

We have, for $n \geq 1$,

$$-x(t) = y(t) - z(t) \leq y(v_n -) - \varepsilon, \quad \forall t \in I_n.$$

This yields

$$\begin{aligned} y(v_n -) &= \max\left(y(u_n), \sup_{u_n < t < v_n} \max(-x(t), 0)\right) \\ &\leq \max(y(u_n), y(v_n -) - \varepsilon), \end{aligned}$$

which means that $y(v_n -) = y(u_n)$, for every $n \geq 1$. It follows that

$$\int_0^\infty \mathbf{1}_{\{z(t) > \varepsilon\}} dy(t) = 0, \quad \forall n \geq 1.$$

Letting $\varepsilon \rightarrow 0$, we obtain the result.

For the uniqueness part, we consider two solutions, (z, y) and (z', y') , of the Skorohod problem with input function x . Then,

$$z(t) - z'(t) = y(t) - y'(t), \quad \forall t \geq 0.$$

Integrating by parts, this yields that for every $t > 0$,

$$\begin{aligned} &[(z - z')(t)]^2 \\ &= 2 \int_{(0, t]} (z - z')(s) d(y - y')(s) - \sum_{0 < s \leq t} [(z - z')(s) - (z - z')(s-)]^2 \\ &\leq 2 \int_{(0, t]} (z - z')(s) d(y - y')(s). \end{aligned}$$

On the other hand, the relations

$$\int_0^\infty z(s) dy(s) = 0 \quad \text{and} \quad \int_0^\infty z'(s) dy'(s) = 0$$

imply that

$$\int_{(0,t]} (z - z')(s) d(y - y')(s) \leq 0, \quad \forall t \geq 0.$$

Hence $z(t) = z'(t)$, $\forall t \geq 0$, which proves the result. \square

Now suppose that M is a symmetric stable process of arbitrary index α defined on a probability space (Ω, \mathcal{F}, P) and $x_0 \geq 0$. For all $t \geq 0$ we put

$$K_t := \sup_{0 \leq s \leq t} \max(-M_s - x_0, 0) \quad (5)$$

and let

$$\bar{M}_t := x_0 + M_t + K_t. \quad (6)$$

For all $0 < \alpha \leq 2$, the process M is a right-continuous process with finite left-hand limits. By Lemma 1, the constructed process \bar{M} is a reflected symmetric stable process in the sense of Definition 2. Reflected symmetric α -stable processes were already introduced and studied by S. Watanabe [22].

We can also regard the symmetric α -stable process M as a (strong) Markov process defined on a family $(\Omega, \mathcal{F}, \mathbb{F}, P_x, x \in \mathbb{R})$ of filtered probability spaces such that $P_x(\{M_0 = x\}) = 1$ for every $x \in \mathbb{R}$. Then, as it is noticed in [22], the reflected process is a strong Markov process on $[0, +\infty)$ when viewed as

$$\bar{M}_t := M_t + \sup_{0 \leq s \leq t} \max(-M_s, 0), \quad t \geq 0.$$

(The difference from (5) and (6) is due to the fact that in this framework M does not necessarily start at 0.)

Let us consider the following measure on $[0, \infty)$:

$$m(dy) := n(y)dy, \quad y > 0, \quad (7)$$

where $n(y) := \frac{\alpha}{2}y^{\frac{\alpha}{2}-1}$. Then the process \bar{M} has m as its invariant measure ([22]), which means that for every Borel measurable set $A \subseteq [0, \infty)$,

$$\int_0^\infty P_x(\bar{M}_t \in A) m(dx) = m(A), \quad \forall t \geq 0. \quad (8)$$

For $\alpha = 2$, m becomes exactly the Lebesgue measure on the interval $[0, \infty)$.

In order to discuss the recurrence properties of the process \bar{M} , let us remind some standard concepts for Markov processes. First, for any, say, standard Markov process X with state space (E, \mathcal{E}) , defined on the corresponding family of probability spaces $(\Omega, \mathcal{F}, \mathbb{F}, P_x, x \in E)$, we introduce the so-called *potential-measures* $U(x, \cdot)$ as

$$U(x, A) := \mathbf{E}_x \left(\int_0^\infty \mathbf{1}_{\{X_t \in A\}} dt \right), \quad A \in \mathcal{E}.$$

Then X is called *recurrent* if, for every measurable set A , $U(\cdot, A) \equiv \infty$ or $U(\cdot, A) \equiv 0$. In [2], the definition of recurrence is given using *nearly measurable* sets, but it is immediately seen that these definitions are equivalent.

Proposition 1. *For all $0 < \alpha \leq 2$, the process \bar{M} is recurrent.*

Proof. From the construction of the process \bar{M} it follows that it returns into the origin at arbitrarily large times. Indeed, let us consider the stopping times

$$\tau_z := \inf \{t \geq 0 : M_t \leq -z\}, \quad z \geq 0. \quad (9)$$

Then, P_x -a.s., $\tau_n < +\infty$, $\forall n \in \mathbb{N}$ and $\tau_n \nearrow +\infty$ as $n \rightarrow +\infty$, a consequence of the fact that $\liminf_{t \rightarrow +\infty} M_t = -\infty$ (see, for example p. 222, [1]) and of the boundedness of M on finite intervals. It is obvious that, for every $n \geq 0$, $\bar{M}_{\tau_n} = 0$, P_x -a.s.

Let U be the potential measure associated with \bar{M} and suppose that there exist $z \geq 0$ and a Borel set A such that $U(z, A) < +\infty$. All we have to prove is that $U(x, A) = 0$, for all $x \geq 0$. First, we show that $\mu(A) = 0$, where μ is the Lebesgue measure on the positive half-line.

By the strong Markov property, for every $n \in \mathbb{N}$ we have

$$\mathbf{E}_x \left(\int_{\tau_n}^{\infty} \mathbf{1}_{\{\bar{M}_t \in A\}} dt \right) = \mathbf{E}_0 \left(\int_0^{\infty} \mathbf{1}_{\{\bar{M}_t \in A\}} dt \right), \quad \forall x \geq 0.$$

For the particular choice $x = z$, passing to the limit as $n \rightarrow \infty$, the finiteness of the left-hand term in this equality implies that $U(0, A) = 0$ and so

$$\int_{\tau_0}^{\infty} \mathbf{1}_{\{\bar{M}_t \in A\}} dt = 0, \quad P_x\text{-a.s.}, \quad \forall x \geq 0. \quad (10)$$

On the other hand, relation (8) yields

$$t m(A) = \int_0^{\infty} \mathbf{E}_x \left(\int_0^t \mathbf{1}_{\{\bar{M}_s \in A\}} ds \right) m(dx).$$

From (10) and the property that $M_s = \bar{M}_s$ on $\{s < \tau_0\}$, we get

$$t m(A) = \int_0^{\infty} \int_0^t P_x(M_s \in A, s < \tau_0) ds m(dx), \quad \forall t \geq 0. \quad (11)$$

The measure $P_x(M_t \in dy, t < \tau_0)$ is the transition function of the process M starting at x which is killed as soon as it leaves $(0, +\infty)$. D. Ray [11] proved that it is absolutely continuous with respect to the Lebesgue measure and its density, $\tilde{p}(t, x, y)$ satisfies

$$\int_0^{\infty} \tilde{p}(t, x, y) dt = \frac{1}{\Gamma(\alpha/2)^2} \int_0^{\min(x, y)} \xi^{\frac{\alpha}{2}-1} (\xi + |y - x|)^{\frac{\alpha}{2}-1} d\xi, \quad x, y \geq 0,$$

where Γ is the Gamma function. We also denote by $p(s, x, y)$ the density of the measure $P_x(M_s \in dy)$. Without loss of generality, we can assume that the set A is bounded. We choose $a > 0$ such that $A \subseteq [0, a]$, and thus $\frac{\alpha}{2} a^{\frac{\alpha}{2}-1} \mu(A) \leq m(A)$. Splitting the integral in (11), we obtain, for every $t \geq 0$,

$$\begin{aligned}
tm(A) &\leq \int_0^a \int_0^\infty \int_A \tilde{p}(s, x, y) dy ds m(dx) + \int_a^\infty \int_0^t \int_A p(s, x, y) dy ds m(dx) \\
&\leq \frac{1}{\Gamma(\alpha/2)^2} \int_0^a \int_0^a \int_0^{\min(x,y)} \xi^{\frac{\alpha}{2}-1} (\xi + |y-x|)^{\frac{\alpha}{2}-1} d\xi dy m(dx) \\
&\quad + \frac{\alpha}{2} a^{\frac{\alpha}{2}-1} \int_0^t \int_A \int_y^\infty p(s, x, y) dx dy ds \\
&\leq \frac{1}{\Gamma(\alpha/2)^2} \int_0^a \int_0^a x^{\alpha-1} |y-x|^{\frac{\alpha}{2}-1} dy dx + \frac{\alpha}{4} a^{\frac{\alpha}{2}-1} t \mu(A).
\end{aligned}$$

The last inequality comes from the property that p is homogeneous and symmetric; indeed, we have

$$\int_y^\infty p(s, x, y) dx = \int_y^\infty p(s, 0, y-x) dx = \int_0^\infty p(s, 0, -x) dx = \frac{1}{2},$$

since $p(s, 0, -x) = p(s, 0, x)$ for all $s, x > 0$. Hence

$$\frac{1}{2}tm(A) \leq \frac{1}{\Gamma(\alpha/2)^2} \int_0^a \int_0^a x^{\alpha-1} |y-x|^{\frac{\alpha}{2}-1} dy dx \leq \frac{4a^{\frac{3\alpha}{2}}}{\alpha^2 \Gamma(\alpha/2)^2}.$$

This proves that $m(A) = 0$, t being taken arbitrarily. Therefore, $\mu(A) = 0$.

From (10), the fact that $P_x(M_t \in \cdot, t < \tau_0)$ is absolutely continuous with respect to μ and the relation

$$U(x, A) = \int_0^\infty P_x(M_s \in A, s < \tau_0) ds + \mathbf{E}_x \left(\int_{\tau_0}^\infty \mathbf{1}_{\{M_s \in A\}} ds \right), \quad \forall x \geq 0,$$

we conclude that $U(x, A) = 0$ for all $x \geq 0$. \square

3 Integral Functionals of Reflected Symmetric Stable Processes

Let \bar{M} be a reflected symmetric stable process of index α given by (5) and (6) on a probability space (Ω, \mathcal{F}, P) , with arbitrary initial state $x_0 \geq 0$. For an arbitrary measurable function $f : [0, \infty) \rightarrow [0, \infty]$ we consider the following integral functional:

$$T_t := \int_0^t f(\bar{M}_s) ds, \quad t \geq 0. \quad (12)$$

The first problem we analyse is whether the functional (12) is finite for all $t > 0$.

For every $y \geq 0$, let us denote by $\mathcal{U}(y)$ the family of open neighborhoods in $[0, +\infty)$ of y , and introduce

$$E_f := \{y \geq 0 : \int_U f(z)m(dz) = +\infty, \forall U \in \mathcal{U}(y)\},$$

where m is the measure introduced by (7). We write $f \in L^{\text{loc}}(m)$ to denote that f is *locally integrable* with respect to the measure m , i.e., $\int_C f(z)m(dz) < \infty$ for every compact subset C of $[0, \infty)$ (which is equivalent to $E_f = \emptyset$).

We remind that a measurable set A is called *polar* if $P(D(A) = +\infty) = 1$, where $D(A) := \inf\{t > 0 : \bar{M}_t \in A\}$ is the *first hitting time* of the set A by the process \bar{M} .

If $0 < \alpha \leq 1$, we define the function $h_{\alpha, x_0} : [0, \infty) \rightarrow [0, \infty]$ by

$$h_{\alpha, x_0}(y) := \begin{cases} |y - x_0|^{\alpha-1}, & 0 < \alpha < 1, \\ |\ln |y - x_0||, & \alpha = 1, \end{cases}$$

and we assume the following hypothesis:

$$(H_{\alpha, x_0}) \quad \begin{cases} E_f \text{ is polar} \\ \text{and } \exists U \in \mathcal{U}(x_0) : \int_U h_{\alpha, x_0}(z) f(z) dz < +\infty, & 0 < \alpha \leq 1, \\ f \in L^{\text{loc}}(m), & 1 < \alpha < 2. \end{cases}$$

Remark. Of course, since the polarity of a set depends only on the law of the considered process, condition (H_{α, x_0}) will depend only on α , x_0 and f . In the case $x_0 = 0$, $0 < \alpha \leq 1$, if E_f is polar, then the condition $\int_U h_{\alpha, x_0}(z) f(z) dz < +\infty$ for some $U \in \mathcal{U}(x_0)$ is automatically satisfied. Indeed, if E_f is polar then 0 cannot belong to E_f (cf. beginning of the proof of Proposition 1). Hence, there exists $U \in \mathcal{U}(0)$ such that

$$\int_U f(z)m(dz) < +\infty.$$

This yields

$$\int_U h_{\alpha, 0}(z) f(z) dz < +\infty.$$

Theorem 1. *Let $\alpha \in (0, 2]$ and $x_0 \geq 0$. Suppose that f satisfies condition (H_{α, x_0}) . Then we have*

$$T_t < +\infty \text{ for all } t \geq 0, P\text{-a.s.}$$

Proof. First of all we note that the set E_f is closed. Hence we can find an increasing sequence Q_N of open sets (in $[0, \infty)$) with compact closure $\bar{Q}_N \subset E_f^c$ such that $E_f^c = \bigcup_{N=1}^{\infty} Q_N$. We introduce the following sequence of stopping times:

$$\rho_N := \inf\{t \geq 0 : \bar{M}_t \in Q_N^c\}.$$

It is easy to see that f is integrable over Q_N with respect to m . The quasi-left continuity of the process M implies the quasi-left continuity of \bar{M} . From this, the fact that E_f is polar and $x_0 \in E_f^c$, one can conclude that ρ_N increases to infinity as $N \rightarrow \infty$, P -a.s.

We define the stopping time

$$\sigma := \inf \{t \geq 0 : M_t \leq -x_0\}.$$

Then

$$\begin{aligned} \int_0^{t \wedge \rho_N} f(\bar{M}_s) ds &= \int_0^{t \wedge \rho_N \wedge \sigma} f(x_0 + M_s) ds + \int_{t \wedge \rho_N \wedge \sigma}^{t \wedge \rho_N} f(\bar{M}_s) ds \\ &\leq \int_0^t \mathbf{1}_{Q_N}(x_0 + M_s) f(x_0 + M_s) ds \\ &\quad + e^t \int_\sigma^\infty e^{-s} \mathbf{1}_{Q_N}(\bar{M}_s) f(\bar{M}_s) ds. \end{aligned} \quad (13)$$

But the function $\mathbf{1}_{Q_N}(x_0 + \cdot) f(x_0 + \cdot)$ is integrable on the real line with respect to the Lebesgue measure. Then, condition (H_{α, x_0}) implies that the assumptions of Corollary 2.2, Proposition 2.5, and Proposition 2.7 from [5] in the cases $\alpha > 1$, $\alpha = 1$ and $\alpha < 1$, respectively, are fulfilled. Therefore,

$$\int_0^t \mathbf{1}_{Q_N}(x_0 + M_s) f(x_0 + M_s) ds < +\infty. \quad (14)$$

We now deal with the other term of the right-hand side of the inequality (13). Let

$$\eta_\lambda(x) := \frac{2}{\alpha \Gamma(\alpha/2) \Gamma(1 + (\alpha/2))} \left(n(x) - \lambda \int_0^\infty \bar{g}_\lambda(x, y) m(dy) \right), \quad \lambda > 0, \quad (15)$$

where \bar{g}_λ denotes the Green function of the resolvent operator corresponding to \bar{M} .

Lemma 2. *For any $\lambda > 0$ and any positive Borel measurable function g , it holds*

$$\mathbf{E} \left(\int_\sigma^\infty e^{-\lambda t} g(\bar{M}_t) dt \right) = \frac{\Gamma(1 + (\alpha/2))}{\sqrt{\lambda}} \mathbf{E} (e^{-\lambda \sigma}) \int_0^\infty \eta_\lambda(y) g(y) dy. \quad (16)$$

The formula (16) was proven in [22] (see the proofs of Theorem 5.2 and Theorem 5.3) for a reflected symmetric stable process \bar{M} defined on $(-\infty, 0]$. According to the symmetry of a symmetric stable process, the behavior of the process M on $[0, \infty)$ is the same as the behavior of $-M$ on $(-\infty, 0]$. Using this, the proof of Lemma 2 follows the same steps as the proof in [22], and we omit the details.

Using the nonnegativity of the function $\bar{g}_\lambda(x, y)$ for all (x, y) , relation (15), and choosing $\lambda = 1$ and $g = \mathbf{1}_{Q_N} f$, we obtain

$$\begin{aligned} \mathbf{E} \int_{\sigma}^{\infty} e^{-s} \mathbf{1}_{Q_N}(\bar{M}_s) f(\bar{M}_s) ds &\leq \Gamma\left(\frac{\alpha}{2} + 1\right) \int_{Q_N} \eta_1(y) f(y) dy \\ &\leq \frac{2}{\alpha \Gamma(\alpha/2)} \int_{Q_N} f(y) m(dy) < +\infty. \end{aligned} \quad (17)$$

Relations (14) and (17) yield that for all $t \geq 0$ and $N \geq 1$, $T_{t \wedge \rho_N} < +\infty$, P -a.s., which proves the theorem. \square

Example 1. The aim of this example is to show that, in the case $1 < \alpha \leq 2$,

$$\int_0^t (\bar{M}_s)^{-\beta} ds = +\infty, \quad \forall t > 0, \quad P\text{-a.s.} \quad (18)$$

if $\beta \geq \alpha/2$ and $x_0 = 0$. This indicates that the condition $f \in L^{\text{loc}}(m)$ seems to be optimal for the convergence of the integral functionals $\int_0^t f(\bar{M}_s) ds$, $t > 0$.

For $1 < \alpha \leq 2$, it is well-known that the symmetric α -stable process M has a local time $L^M(t, a)$, jointly continuous in (t, a) (cf., e.g., [3]). It is then natural to ask whether that still holds in the case of the reflected process \bar{M} .

For any Borel set A , let

$$S(t, A) := \int_0^t \mathbf{1}_A(\bar{M}_s) ds, \quad t \geq 0,$$

denote the sojourn time of \bar{M} in A . In [9], F. B. Knight proved that for every $\alpha \in (0, 2]$ there exists the local time in 0 of \bar{M} , which we denote $L^{\bar{M}}(t, 0)$. In the case $x_0 = 0$, the following holds:

$$0 < L^{\bar{M}}(t, 0) = \lim_{\varepsilon \searrow 0} \varepsilon^{-\alpha/2} S(t, [0, \varepsilon]), \quad \forall t > 0, \quad P\text{-a.s.} \quad (19)$$

Suppose now that $1 < \alpha \leq 2$. For every $\omega \in \Omega$, the set $\{t \geq 0 : \bar{M}_t(\omega) > 0\}$ can be written as the union of pairwise disjoint intervals of the form $I_n(\omega) = (u_n(\omega), v_n(\omega))$ or $[u_n(\omega), v_n(\omega))$, $n \geq 1$. As shown in the proof of Lemma 1, the reflecting force $K(\omega)$ is constant on I_n . We denote this constant by $k_n(\omega)$. If $a > 0$ we define

$$L^{\bar{M}}(t, a) := \sum_{n=1}^{\infty} (L^M(v_n \wedge t, a - x_0 - k_n) - L^M(u_n \wedge t, a - x_0 - k_n)), \quad t \geq 0. \quad (20)$$

We give a concise proof of the *occupation times formula* for $L^{\bar{M}}$, i.e.,

$$\int_0^t g(\bar{M}_s) ds = \int_0^{\infty} g(a) L^{\bar{M}}(t, a) da, \quad \forall t \geq 0,$$

for every positive measurable function g , P -a.s. By (19) and the strong Markov property of \bar{M} (when considered as such),

$$\mu(\{t \geq 0 : \bar{M}_t(\omega) = 0\}) = 0, \quad P\text{-a.s.}$$

(recall that μ denotes the Lebesgue measure on the positive half-line). Therefore it is sufficient to show the occupation times formula only for the function of the type $g \equiv \mathbf{1}_A$, where A is a Borel set in $(0, \infty)$. Integrating with respect to $a \in A$ in (20) and using the occupation times formula for L^M , we obtain

$$\begin{aligned} \int_A L^{\bar{M}}(t, a) da &= \sum_{n=1}^{\infty} \int_{A-x_0-k_n} (L^M(v_n \wedge t, a) - L^M(u_n \wedge t, a)) da \\ &= \sum_{n=1}^{\infty} \int_{u_n \wedge t}^{v_n \wedge t} \mathbf{1}_{A-x_0-k_n}(M_s) ds = \sum_{n=1}^{\infty} \int_{u_n \wedge t}^{v_n \wedge t} \mathbf{1}_A(\bar{M}_s) ds \\ &= \int_0^{\infty} \mathbf{1}_A(\bar{M}_s) ds, \quad t \geq 0. \end{aligned}$$

That means $L^{\bar{M}}(t, a)$ is a possible candidate for the local time of the process \bar{M} .

The equality (18) is then a consequence of the occupation times formula. Indeed, from this it follows

$$\int_0^t (\bar{M}_s)^{-\beta} ds = \int_0^{\infty} a^{-\beta} L^{\bar{M}}(t, a) da, \quad \forall t \geq 0, \quad P\text{-a.s.},$$

and

$$S(t, [0, \varepsilon)) = \int_0^{\varepsilon} L^{\bar{M}}(t, a) da, \quad \forall t, \varepsilon > 0, \quad P\text{-a.s.}$$

Using integration by parts and (19), for sufficiently small $\varepsilon > 0$, we obtain

$$\begin{aligned} \int_0^t (\bar{M}_s)^{-\beta} ds &\geq \int_{\varepsilon}^1 a^{-\beta} L^{\bar{M}}(t, a) da \geq \int_{\varepsilon}^1 a^{-\alpha/2} L^{\bar{M}}(t, a) da \\ &= S(t, [0, 1)) - \varepsilon^{-\alpha/2} S(t, [0, \varepsilon)) + \frac{\alpha}{2} \int_{\varepsilon}^1 a^{-\frac{\alpha}{2}-1} S(t, [0, a)) da \\ &\geq -\varepsilon^{-\alpha/2} S(t, [0, \varepsilon)) + \frac{\alpha}{4} L^{\bar{M}}(t, 0) \int_{\varepsilon}^{\sqrt{\varepsilon}} a^{-1} da \\ &\geq \frac{1}{8} L^{\bar{M}}(t, 0) (-\alpha \ln \varepsilon - 16). \end{aligned}$$

The result then follows by letting $\varepsilon \rightarrow 0$.

Of course, if $\beta < \alpha/2$, Theorem 1 ensures us that $\int_0^t (\bar{M}_s)^{-\beta} ds < \infty$, $\forall t \geq 0$. This means that $\alpha/2$ is the critical exponent for the convergence or the divergence of the integral.

Let us now discuss briefly the conditions ensuring that

$$T_{\infty} = \int_0^{\infty} f(\bar{M}_s) ds = \infty.$$

Because the process \bar{M} is a recurrent one, it is logically to expect to have similar sufficient conditions found for the symmetric stable process of index $1 < \alpha \leq 2$, e.g. see [5].

Theorem 2. *Suppose that $\mu(\{a : f(a) > 0\}) > 0$. Then, for all $x_0 \geq 0$ and $\alpha \in (0, 2]$, $T_\infty = \infty$, P -a.s.*

Proof. For the convenience of the reader, we give a proof which is slightly different of that of Proposition 2.6 [5].

It is sufficient to prove the assertion in the case $f = \mathbf{1}_A$, where A is an arbitrary Borel measurable set with $\mu(A) > 0$.

Let $\varepsilon > 0$. By the strong Markov property of \bar{M} , we have that for every $n \in \mathbb{N}$

$$\begin{aligned} P_0 \left(\int_0^\infty \mathbf{1}_A(\bar{M}_s) ds > \varepsilon \right) &= P_{\bar{M}_{\tau_n}} \left(\int_0^\infty \mathbf{1}_A(\bar{M}_s) ds > \varepsilon \right) \\ &= P_{x_0} \left(\left\{ \int_{\tau_n}^\infty \mathbf{1}_A(\bar{M}_s) ds > \varepsilon \right\} \middle| \mathcal{F}_{\tau_n} \right), \quad P_{x_0}\text{-a.s.}, \end{aligned}$$

with τ_n defined by (9). The fact that $\lim_{n \rightarrow \infty} \tau_n = +\infty$, P_{x_0} -a.s., allows us to pass to the limit as $n \rightarrow \infty$ in this relation; from the theorem of Lebesgue-Lévy on the convergence of conditional expectations we obtain that

$$P_0 \left(\int_0^\infty \mathbf{1}_A(\bar{M}_s) ds > \varepsilon \right) = \mathbf{1}_{\bigcap_{n \in \mathbb{N}} \{ \int_{\tau_n}^\infty \mathbf{1}_A(\bar{M}_s) ds > \varepsilon \}}, \quad P_{x_0}\text{-a.s.}$$

In the proof of Proposition 1, we have shown that $\mu(A) > 0$ implies $U(0, A) = +\infty$; thus $P_0 \left(\int_0^\infty \mathbf{1}_A(\bar{M}_s) ds > \varepsilon \right) > 0$. Consequently,

$$P_{x_0} \left(\bigcap_{n \in \mathbb{N}} \left\{ \int_{\tau_n}^\infty \mathbf{1}_A(\bar{M}_s) ds > \varepsilon \right\} \right) = 1.$$

Using once again the unboundedness of the sequence $(\tau_n)_{n \in \mathbb{N}}$, this relation gives

$$\int_0^\infty \mathbf{1}_A(\bar{M}_s) ds = +\infty, \quad P_{x_0}\text{-a.s.},$$

which proves Theorem 2. □

4 Existence and Uniqueness of Solutions

Let us consider a Borel measurable function $b : [0, \infty) \rightarrow \mathbb{R}$, an arbitrary initial value $x_0 \geq 0$ and an arbitrary stability index $\alpha \in (0, 2]$. We also define the set

$$N_b := \{x \geq 0 : b(x) = 0\}.$$

Theorem 3. *Assume that $|b|^{-\alpha}$ satisfies condition (H_{α, x_0}) . Then there exists a (non-exploding) solution X of (1) with $X_0 = x_0$. For this solution, the property*

$$\int_0^\infty \mathbf{1}_{N_b}(X_s) ds = 0, \quad P\text{-a.s.}$$

is satisfied.

Proof. On a stochastic basis $(\Omega, \mathcal{F}, P; \mathbb{F})$ we consider a symmetric α -stable process (M^*, \mathbb{F}) and the corresponding reflected process \bar{M}^* , defined by

$$\begin{aligned} K_t^* &:= \sup_{0 \leq s \leq t} \max(-M_s^* - x_0, 0); \\ \bar{M}_t^* &:= x_0 + M_t^* + K_t^*. \end{aligned} \quad (21)$$

Let

$$T_t = \int_0^t |b|^{-\alpha}(\bar{M}_s^*) ds, \quad t \geq 0, \quad (22)$$

and

$$A_t = \inf\{s \geq 0 : T_s > t\}.$$

It follows from Theorem 1 that T is a P -a.s. finite and continuous \mathbb{F} -adapted process with $T_0 = 0$. Clearly, the condition of Theorem 2 is satisfied because $|b|^{-\alpha}$ is strictly positive. (Note that $|b|^{-\alpha}(x) = +\infty$ if $b(x) = 0$). Consequently, $T_\infty = \infty$. Due to its definition, the process A is then a right-continuous \mathbb{F} -time change defined for all $t \in [0, \infty)$. The condition $T_\infty = \infty$ means that $A_t < \infty$ for all $t > 0$, and we have $A_\infty := \lim_{t \rightarrow \infty} A_t = \infty$, since T is finite. Moreover, the process A is continuous on $[0, \infty)$ because T is strictly increasing, which is also a consequence of the strict positivity of $|b|^{-\alpha}$. One can easily check that $A = T^{-1}$.

On the other side, the process \bar{M}^* is a right-continuous semimartingale because M^* is a right-continuous semimartingale and K^* is a right-continuous and increasing process. Then due to the well-known time change theorem for semimartingales (see, e.g., [6], Theorem 10.16), the process (X, \mathbb{G}) , where

$$X_t := \bar{M}_{A_t}^*, \quad \mathcal{G}_t := \mathcal{F}_{A_t}, \quad t \geq 0, \quad (23)$$

is again a right-continuous semimartingale. From (21) we then have

$$X_t = x_0 + M_{A_t}^* + K_{A_t}^*, \quad t \geq 0.$$

Obviously, $X_t \geq 0$ for all $t \geq 0$. We put $\tilde{M}_t = M_{A_t}^*$ and $K_t = K_{A_t}^*$.

Lemma 3. *K is a reflecting force for X , i.e., K is increasing, right-continuous, $K_0 = 0$ and*

$$\int_0^\infty \mathbf{1}_{\{X_s \neq 0\}} dK_s = 0. \quad (24)$$

Proof. It follows directly from the definition of the process K^* and the continuity of A that K is also a right-continuous and increasing process with $K_0 = 0$. Moreover, for every $t \geq 0$, it holds

$$\begin{aligned} K_t &= \sup_{0 \leq s \leq A_t} \max(-M_s^* - x_0, 0) \\ &= \sup_{0 \leq s \leq t} \max(-M_{A_s}^* - x_0, 0) \\ &= \sup_{0 \leq s \leq t} \max(-\tilde{M}_s - x_0, 0) . \end{aligned}$$

Consequently, we have

$$X_t = x_0 + \tilde{M}_t + \sup_{0 \leq s \leq t} \max(-\tilde{M}_s - x_0, 0) ,$$

and from Lemma 1 it follows that the relation (24) is true. \square

Lemma 4. *It holds*

$$\mathbf{E} \left(\int_0^t \mathbf{1}_{N_b}(\bar{M}_{s-}^*) ds \right) = \mathbf{E} \left(\int_0^t \mathbf{1}_{N_b}(\bar{M}_s^*) ds \right) = 0 , \quad \forall t \geq 0 .$$

Proof. Using Lemma 2 for the function $g = \mathbf{1}_{N_b \setminus E_{|b|^{-\alpha}}}$ and $\lambda = 1$, we estimate

$$\begin{aligned} &\mathbf{E} \left(\int_0^t g(\bar{M}_s^*) ds \right) \\ &\leq \mathbf{E} \left(\int_0^{\sigma \wedge t} g(x_0 + M_s^*) ds \right) + e^t \mathbf{E} \left(\int_\sigma^\infty e^{-s} g(\bar{M}_s^*) dt \right) \\ &\leq \int_0^t P \left(x_0 + M_s^* \in N_b \setminus E_{|b|^{-\alpha}} \right) ds + \frac{2e^t}{\alpha \Gamma(\alpha/2)} \int_0^\infty g(y) m(dy) . \end{aligned}$$

It is obvious that $m(N_b \setminus E_{|b|^{-\alpha}}) = 0$. The right-hand side is then equal to zero due to the equivalence between the Lebesgue measure and m , on one hand, and to the absolute continuity of the distribution of M_s^* , on the other hand. The polarity of $E_{|b|^{-\alpha}}$ is used in order to finish the proof. \square

Lemma 5. *There exists a symmetric stable process M of the same index α such that for all $t \geq 0$ we have*

$$\tilde{M}_t = \int_0^t b(X_{s-}) dM_s . \quad (25)$$

Proof. Because we have $T_t < \infty$ for every $t \geq 0$ and the integrand $|b|^{-\alpha}(\bar{M}_t^*)$ is \mathcal{F}_t -measurable, we can conclude that for all $t \geq 0$ there exists the stochastic integral $\int_0^t b^{-1}(\bar{M}_{s-}^*) dM_s^*$ (cf. [13], [7]). On the other side, from the time change properties in stochastic integrals with respect to stable processes (cf., e.g., [5] or [13]) it follows that the process M , defined by

$$M_t := \int_0^{A_t} b^{-1}(\bar{M}_{s-}^*) dM_s^*, \quad t \geq 0, \quad (26)$$

is a \mathbb{G} -adapted symmetric α -stable process.

A simple use of Lemma 4 shows that

$$A_t = \int_0^{A_t} \mathbf{1}_{N_b^c}(\bar{M}_s^*) ds = \int_0^{A_t} |b|^\alpha(\bar{M}_s^*) |b|^{-\alpha}(\bar{M}_s^*) ds, \quad t \geq 0.$$

Consequently, changing the variables in the Lebesgue-Stieltjes integral, from relation (22) and $A = T^{-1}$ we obtain

$$A_t = \int_0^{A_t} |b|^\alpha(\bar{M}_s^*) dT_s = \int_0^{T_{A_t}} |b|^\alpha(\bar{M}_{A_s}^*) ds = \int_0^t |b|^\alpha(X_s) ds, \quad t \geq 0.$$

Therefore, with similar arguments as above, we have that there exists the stochastic integral

$$\int_0^t b(X_{s-}) dM_s, \quad t \geq 0. \quad (27)$$

Now, using time change properties for stochastic integrals with respect to semimartingales (see, e.g. [6], Chap. X) and taking into account (26) and (27), we obtain

$$M_t = \int_0^{A_t} b^{-1}(\bar{M}_{s-}^*) dM_s^* = \int_0^t b^{-1}(X_{s-}) dM_{A_s}^*, \quad t \geq 0,$$

and, consequently,

$$\int_0^t b(X_{s-}) dM_s = \int_0^t b(X_{s-}) b^{-1}(X_{s-}) dM_{A_s}^*, \quad t \geq 0. \quad (28)$$

From Lemma 4 we can conclude

$$\int_0^t \mathbf{1}_{N_b}(\bar{M}_{s-}^*) dM_s^* = 0, \quad \forall t \geq 0, \quad P\text{-a.s.},$$

which implies that

$$\int_0^t b(\bar{M}_{s-}^*) b^{-1}(\bar{M}_{s-}^*) dM_s^* = M_t^*, \quad \forall t \geq 0, \quad P\text{-a.s.}$$

Using once again the properties of time change for stochastic integrals, we have

$$\int_0^t b(X_{s-}) b^{-1}(X_{s-}) dM_{A_s}^* = M_{A_t}^*, \quad \forall t \geq 0, \quad P\text{-a.s.}$$

Combined with (28), this relation yields (25). \square

We have shown that the process X has the form

$$X_t = x_0 + \int_0^t b(X_{s-}) dM_s + K_t, \quad t \geq 0,$$

where K is the reflecting force for X . Therefore, X is a solution of (1). From Lemma 4 one can easily conclude that X also satisfies

$$\int_0^\infty \mathbf{1}_{N_b}(X_s) ds = 0, \quad P\text{-a.s.}$$

This completes the proof of Theorem 3. \square

Remark. For $\alpha = 2$ the assumption $|b|^{-\alpha} \in L^{\text{loc}}(m)$ reduces to the condition that b^{-2} is locally integrable over the half-line $[0, \infty)$, which coincides with the condition found by W. M. Schmidt [16] for the case of a Brownian motion.

Finally we investigate the uniqueness in law of the solution of (1). At first we notice that, in general, condition (H_{α, x_0}) does not ensure the uniqueness in law of the solution. We give the following general, but very simple example.

Example 2. Let the volatility b be such that $|b|^{-\alpha}$ satisfies condition (H_{α, x_0}) . Suppose that $b(x_0) = 0$. Then the solution X of (1) with $X_0 = x_0$ is not unique in law. Indeed, according to Theorem 3, there is a solution X of (1) such that

$$\int_0^\infty \mathbf{1}_{N_b}(X_s) ds = 0, \quad P\text{-a.s.}$$

On the other side, we may put $Y \equiv x_0$; obviously, Y is a solution of (1) with $Y_0 = x_0$. It is clear that X and Y have different laws.

This example can be generalized as follows. Suppose that $|b|^{-\alpha}$ satisfies condition (H_{α, x_0}) . Let X be the solution of (1) constructed in the proof of Theorem 3.

We assume that the *first entry time* $D(N_b)$ of X into N_b ,

$$D(N_b) := \inf \{t \geq 0 : b(X_t) = 0\},$$

is finite with positive probability:

$$P(D(N_b) < +\infty) > 0.$$

Then the process Y obtained by stopping X at $D(N_b)$,

$$Y_t := X_{t \wedge D(N_b)}, \quad t \geq 0,$$

is again a solution of (1) with $Y_0 = x_0$ and, obviously, the laws of X and Y are different. This motivates the following

Definition 3. Let condition $(H_{x_0, \alpha})$ for $|b|^{-\alpha}$ be satisfied. A solution X of (1) is called a *fundamental solution* if it holds

$$\int_0^\infty \mathbf{1}_{N_b}(X_s) ds = 0, \quad P\text{-a.s.} \quad (29)$$

It is natural to expect that the solution X of (1) with $X_0 = x_0$ is unique in law in the class of fundamental solutions. For preparing this result, let X be an arbitrary solution of (1) with $X_0 = x_0$ given on the stochastic basis $(\Omega, \mathcal{F}, P; \mathbb{F})$. We define $A = (A_t)_{t \geq 0}$ by

$$A_t := \int_0^t |b|^\alpha(X_s) ds, \quad t \geq 0. \quad (30)$$

From [7], [13] we know that $A_t < +\infty$, $t \geq 0$, P -a.s. and hence is a P -a.s. finite continuous \mathbb{F}^X -adapted process. We introduce the right inverse $T = (T_t)_{t \geq 0}$ of A :

$$T_t := \inf \{s \geq 0 : A_s > t\}, \quad t \geq 0.$$

By $\mathbb{G} := \mathbb{F}^X \circ T$ we denote the filtration $(\mathcal{F}_{T_t}^X)_{t \geq 0}$. To begin with, we will prove the following representation of the solution X .

Proposition 2. *On a, possibly, enlarged stochastic basis $(\Omega', \mathcal{F}', P'; \mathbb{F}')$ there exists a reflected symmetric stable process \bar{M}^* of index α with $\bar{M}_0^* = x_0$ such that*

$$X_t = \bar{M}_{A_t}^*, \quad t \geq 0, \quad P\text{-a.s.} \quad (31)$$

Proof. We have

$$X_t = x_0 + \int_0^t b(X_{s-}) dM_s + K_t, \quad t \geq 0,$$

where M is a symmetric stable process of index α (with $M_0 = 0$) and K is a reflecting force for X . According to [5] (Proposition 4.3), changing the roles of A and T , the process $\tilde{M}^* = (\tilde{M}_t^*)_{t \geq 0}$ defined by

$$\tilde{M}_t^* := \int_0^{T_t} b(X_{s-}) dM_s, \quad t \geq 0,$$

is a symmetric stable process of index α stopped at $A_{T_\infty-} = A_\infty$ (this latter equality holds because $T_\infty = +\infty$ P -a.s.). Using [5] (Lemma 4.2), we obtain that there exists a symmetric stable process M^* of index α (on a certain extension of $(\Omega, \mathcal{F}, P; \mathbb{G})$) such that

$$\tilde{M}_t^* = M_{t \wedge A_\infty}^*, \quad t \geq 0.$$

We now define the reflected symmetric stable process \bar{M}^* and the reflecting force K^* by (21). In order to verify relation (31), we first remark that, for all $t \geq 0$,

$$M_{A_t}^* = \tilde{M}_{A_t}^* = \int_0^{T_{A_t}} b(X_{s-}) dM_s = \int_0^t b(X_{s-}) dM_s,$$

the latter being true because $[t, T_{A_t}]$ are intervals of constancy for A and hence for $\int_0^\cdot b(X_{s-}) dM_s$ (cf. [5], Proposition 4.3 (iv)). Furthermore, for all $t \geq 0$,

$$\begin{aligned} K_{A_t}^* &= \sup_{0 \leq s \leq A_t} \max(-M_s^* - x_0, 0) = \sup_{0 \leq s \leq t} \max(-M_{A_s}^* - x_0, 0) \\ &= \sup_{0 \leq s \leq t} \max\left(-\int_0^s b(X_{u-}) dM_u - x_0, 0\right) = K_t, \end{aligned}$$

the reflecting force for X , because of Lemma 1. This proves $\bar{M}_{A_t}^* = X_t, t \geq 0$, and hence Proposition 2. \square

Next we give a representation for the increasing process $T = (T_t)_{t \geq 0}$.

Proposition 3. *Suppose that the volatility b is such that $|b|^{-\alpha}$ satisfies condition (H_{α, x_0}) and that X is a fundamental solution of (1) with $X_0 = x_0$. Let \bar{M}^* be a reflected symmetric α -stable process with $\bar{M}_0^* = x_0$ on a, possibly, enlarged stochastic basis, satisfying (31). Then*

$$T_t = \int_0^t |b|^{-\alpha}(\bar{M}_s^*) ds, \quad t \geq 0, \quad P\text{-a.s.} \quad (32)$$

Proof. From (29) and (31), we obtain

$$T_t = \int_0^{T_t} |b|^{-\alpha}(X_s) dA_s = \int_0^{T_t} |b|^{-\alpha}(\bar{M}_{A_s}^*) dA_s$$

and, by time change in this Lebesgue-Stieltjes integral, we get

$$T_t = \int_0^{A_{T_t}} |b|^{-\alpha}(\bar{M}_{A_s}^*) ds.$$

In view of the continuity of A , we conclude $A_{T_t} = t \wedge A_\infty$ and hence

$$T_t = \int_0^{t \wedge A_\infty} |b|^{-\alpha}(\bar{M}_s^*) ds. \quad (33)$$

This yields (32) for $t \leq A_\infty$. In particular,

$$\int_0^{A_\infty} |b|^{-\alpha}(\bar{M}_s^*) ds = T_{A_\infty} = +\infty.$$

From this, we observe that (32) also holds for $t > A_\infty$. \square

The next proposition shows, in particular, that the representation (32) do hold on the same stochastic basis $(\Omega, \mathcal{F}, P; \mathbb{G})$ if $|b|^{-\alpha}$ satisfies condition (H_{α, x_0}) and if X is a fundamental solution. In this case, there is no need for an enlargement.

Proposition 4. *Suppose that $|b|^{-\alpha}$ satisfies condition (H_{α, x_0}) . Let X be a fundamental solution of (1) with $X_0 = x_0$. Then*

$$A_\infty = \int_0^\infty |b|^\alpha(X_s) ds = +\infty \quad P\text{-a.s.}$$

Proof. The assertion is equivalent to

$$T_t < +\infty, \quad t \geq 0, \quad P\text{-a.s.}$$

But this follows from the equality

$$T_t = \int_0^t |b|^{-\alpha} (\bar{M}_s^*) ds, \quad t \geq 0, \quad P\text{-a.s.},$$

cf. Proposition 3 and Theorem 1 for the reflected symmetric α -stable process \bar{M}^* with $\bar{M}_0^* = x_0$ of Proposition 2. \square

Now we turn to the uniqueness in law of the fundamental solution.

Theorem 4. *Suppose that the volatility b is such that $|b|^{-\alpha}$ satisfies condition (H_{α, x_0}) . Then the fundamental solution X of (1) with $X_0 = x_0$ (which exists by Theorem 3) is unique in law. Furthermore,*

$$\int_0^\infty |b|^\alpha (X_s) ds = +\infty \quad P\text{-a.s.}$$

Proof. Let X be a fundamental solution of (1) with $X_0 = x_0$. According to Proposition 2, X is a well-defined measurable functional of (\bar{M}^*, A) , where \bar{M}^* is a reflected symmetric stable process of index α with $\bar{M}_0^* = x_0$. Furthermore, Proposition 3 yields that T , and hence A , is a well-defined measurable functional of \bar{M}^* . Thus we may conclude that X is a well-defined measurable functional of \bar{M}^* . So, the law of X on the Skorohod space is the image law of \bar{M}^* by this measurable mapping and hence uniquely determined. The last statement is exactly the conclusion of Proposition 4. \square

Corollary. *Suppose that $|b|^{-\alpha}$ satisfies condition (H_{α, x_0}) and, moreover,*

$$b(x) \neq 0, \quad \forall x \geq 0.$$

Then the solution X of (1) with $X_0 = x_0$ exists and is unique in law. Furthermore, it holds

$$\int_0^\infty |b|^\alpha (X_s) ds = +\infty \quad P\text{-a.s.}$$

In conclusion, we note that the fundamental solution X of (1) is nothing else than a reflected symmetric α -stable process \bar{M}^* taken in *another, random clock* A given by (30) and satisfying the additional property $A_\infty = +\infty$. In other words, the process X is running through the same trajectories as a reflected symmetric α -stable process but *in different clocks*. So, roughly speaking, the fundamental solution X of (1) has the same recurrence behaviour as a reflected symmetric α -stable process. In particular, X hits the boundary 0 infinitely often P -a.s. Moreover, if $1 < \alpha < 2$, as for $\alpha = 2$, the process X has a local time L^X in the sense of an occupation time density, given by

$$L^X(t, a) = L^{\bar{M}^*}(A_t, a), \quad t \geq 0.$$

Indeed, it can easily be verified that for every nonnegative Borel function g on $[0, +\infty)$

$$\int_0^t g(X_s) dA_s = \int_0^\infty g(a) L^X(t, a) da, \quad t \geq 0, \quad P\text{-a.s.}$$

or, alternatively,

$$\int_0^t g(X_s) ds = \int_0^\infty g(a) L^X(t, a) \mu_b(da), \quad t \geq 0, \quad P\text{-a.s.}$$

where the measure μ_b is given by $\mu_b(da) = |b|^{-\alpha}(a) \mu(da)$.

In the first formula, the occupation time is measured by dA_s , but the occupation time density is taken with respect to the Lebesgue measure μ on $[0, +\infty)$, whereas in the second formula the occupation time is measured by ds , however, in this case, the occupation time density is taken with respect to the new measure μ_b depending on the volatility b .

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