

ON MULTIDIMENSIONAL SDEs WITHOUT DRIFT AND WITH TIME-DEPENDENT DIFFUSION MATRIX

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Abstract

We study multidimensional stochastic equations

$$X_t = x_0 + \int_0^t B(s, X_s) dW_s$$

where x_0 is an arbitrary initial state, W is a d -dimensional Wiener process and $B : [0, +\infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^{d^2}$ is a measurable diffusion coefficient. We give sufficient conditions for the existence of weak solutions. Our main result generalizes some results obtained by A. Rozkosz and L. Słomiński [17] and T. Senf [20] for the existence of weak solutions of one-dimensional stochastic equations and also some results by A. Rozkosz and L. Słomiński [18], [19] for multidimensional equations. Finally, we also discuss the homogeneous case.

Key Words Multidimensional stochastic differential equations, measurable coefficients, diffusion processes, martingales, Wiener process, weak convergence

MR Subject Classification 60H10, 60J60, 60J65, 60G44

1 Introduction

In this article we consider a stochastic equation of the form

$$X_t = x + \int_0^t B(s, X_s) dW_s, \quad t \geq 0, \quad (1)$$

where $B : [0, +\infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^{d^2}$ is a Borel measurable matrix function with $d \geq 1$, W is a d -dimensional Wiener process and $x_0 \in \mathbb{R}^d$ is an arbitrary initial vector. Componentwise written this equation is

$$X_t^i = x_0^i + \sum_{k=1}^d \int_0^t B_{ik}(s, X_s) dW_s^k, \quad i = 1, 2, \dots, d, \quad t \geq 0.$$

It is well-known that if the diffusion coefficient B of the equation satisfies the assumption of at most linear growth then the solution of the equation, if it exists, is nonexploding, i.e., it exists in \mathbb{R}^d for all $t \geq 0$ (cf. [9], Theorem 6.4.2). In the general case the solution exists only in the sense that it may explode, i.e., on a finite time interval it may leave every compact subset of \mathbb{R}^d . Here we study solutions of Eq. (1) in this more general context.

The aim of this article is to give sufficient conditions for the existence of solutions to Eq. (1). A.V. Skorohod [22] was the first who investigated weak solutions of stochastic differential equations with continuous coefficients. N.V. Krylov [12] proved the existence of weak solutions of stochastic equations for only measurable coefficients using his well-known estimates for stochastic integrals of diffusion processes. Together with the boundedness of the drift coefficient, he only assumed that

$$c|z|^2 \leq (B(t, x)z, z), \quad \|B(t, x)\|^2 \leq C, \quad z \in \mathbb{R}^d,$$

for some constants $0 < c < C$ not depending on $(t, x) \in [0, +\infty) \times \mathbb{R}^d$ where (\cdot, \cdot) denotes the Euclidean scalar product and

$$\|B(t, x)\|^2 = \sum_{i,j=1}^d B_{ij}^2(t, x).$$

Using methods of nonstandard analysis, this result was generalized to certain degenerate diffusion matrices by S.A. Kosciuk [11]. Stochastic differential equations with discontinuous coefficients were also considered by S. Anulova and H. Pragarauskas [3].

The case of one-dimensional homogeneous equations (i.e., equations with time-independent coefficients) was treated by the first author and W. Schmidt (cf. [5], [6], [7], [8]). In particular, there was shown that the local integrability of B^{-2} is necessary and sufficient for the existence of nontrivial solutions with an arbitrary initial value.

A more general case with time-dependent coefficients but still with one-dimensional state space was investigated by W. Kurenok [13], A. Rozkosz and L. Słomiński [17] and T. Senf [20]. T. Senf [20] was able to prove that the local integrability of B^2 and B^{-2} ensures the existence of a solution for every initial value and, moreover, the solution does not explode if only, for every $N \geq 1$, there exists a nonnegative function \bar{B}_N finite on a set of positive Lebesgue measure such that $B^2(t, x) \leq \bar{B}_N(x)$ for every $t \in [0, N]$ and $x \in \mathbb{R}$.

Another far-reaching generalization was given by A. Rozkosz and L. Słomiński [18], [19] for multidimensional stochastic equations with time-independent and also with time-dependent diffusion and drift coefficients satisfying, additionally, the usual linear growth condition. In particular, their results also include those of A.S. Kosciuk [11]

In the time-dependent or multidimensional cases, main tools are Krylov's estimates. We also mention the paper by A.V. Melnikov [14] where Krylov's estimates are generalized to continuous semimartingales.

In this article, we restrict ourselves to the multidimensional equation (1) *without drift*. Once we have solved this equation, there are several known methods to treat equations with nonvanishing drift part, too. Instead of the boundedness of B or, more generally, the condition of at most linear growth, we use certain local integrability conditions to ensure the existence of solutions to Eq. (1). The conditions imposed can be improved in the homogeneous case which will briefly be discussed in the final part of the paper.

2 Preliminaries

By $(\bar{\mathbb{R}}^d, \mathcal{B}(\bar{\mathbb{R}}^d))$ we denote the one-point compactification $\bar{\mathbb{R}}^d = \mathbb{R}^d \cup \{\Delta\}$ of \mathbb{R}^d equipped with the σ -algebra $\mathcal{B}(\bar{\mathbb{R}}^d)$ of its Borel subsets. For any function $w : [0, +\infty) \rightarrow \bar{\mathbb{R}}^d$ and $a > 0$, we set

$$\tau_\Delta(w) = \inf\{t \geq 0 : w(t) = \Delta\} \quad \text{and} \quad \tau_a(w) = \inf\{t \geq 0 : |w(t)| \geq a\} \quad (2)$$

and call $\tau_\Delta(w)$ the explosion time of the trajectory w . Let $E([0, +\infty))$ be the set of all right-continuous functions $w : [0, +\infty) \rightarrow \bar{\mathbb{R}}^d$ such that w is continuous on $[0, \tau_\Delta(w))$ and $w(t) = \Delta$ whenever $t \geq \tau_\Delta(w)$. For every $t \geq 0$ we define the coordinate mappings $Z_t : E([0, +\infty)) \rightarrow \bar{\mathbb{R}}^d$ by

$$Z_t(w) = w(t), \quad w \in E([0, +\infty)), \quad (3)$$

and introduce the σ -algebras

$$\mathcal{E}([0, +\infty)) = \sigma(Z_t, t \geq 0) \quad \text{and} \quad \mathcal{E}_t = \sigma(Z_s, s \leq t), \quad t \geq 0,$$

and the filtration $\mathbb{E} = (\mathcal{E}_t)_{t \geq 0}$. Obviously, τ_Δ and, for every $a > 0$, τ_a are \mathbb{E} -stopping times such that $\tau_a(w) < \tau_{a'}(w) < \tau_\Delta(w)$ whenever $\tau_\Delta(w) < +\infty$ and

$a < a'$. We notice that for every \mathbb{I} -stopping time τ the σ -algebra \mathcal{E}_τ associated with τ can be described as

$$\mathcal{E}_\tau = \sigma(Z_{t \wedge \tau}, t \geq 0) = (Z^\tau)^{-1}(\mathcal{E}([0, +\infty))) \quad (4)$$

where $Z^\tau : E([0, +\infty)) \rightarrow E([0, +\infty))$ is defined by $Z^\tau(w) = w(\cdot \wedge \tau)$, $w \in E([0, +\infty))$ (cf. [21], Theorem I.6).

Let now $(a_m)_{m \in \mathbb{N}}$ be an increasing sequence of positive real numbers converging to infinity as $m \rightarrow \infty$. It is known (cf. T. Senf [20]) that, for every $m \in \mathbb{N}$, $(E([0, +\infty)), \mathcal{E}_{\tau_{a_m}})$ is a standard Borel space (cf. K.R. Parthasarathy [16]). Furthermore, it can easily be seen that the σ -algebras $\mathcal{E}_{\tau_{a_m}}$ are increasing, $\mathcal{E}([0, +\infty)) = \sigma(\bigcup_{m \in \mathbb{N}} \mathcal{E}_{\tau_{a_m}})$ and if $(A_m)_{m \in \mathbb{N}}$ is a decreasing sequence of atoms A_m from $\mathcal{E}_{\tau_{a_m}}$ then $\bigcap_{m \in \mathbb{N}} A_m \neq \emptyset$. Theorem V.4.1 of K.R. Parthasarathy [16] now implies

Proposition 2.1 *Let $(\mathbf{Q}^m)_{m \in \mathbb{N}}$ be a consistent family of probability measures \mathbf{Q}^m on $\mathcal{E}_{\tau_{a_m}}$. Then there exists a unique probability measure \mathbf{Q} on $\mathcal{E}([0, +\infty))$ which is an extension of the family $(\mathbf{Q}^m)_{m \in \mathbb{N}}$.*

Let $C([0, +\infty)) \subseteq E([0, +\infty))$ be the space of continuous functions w of $[0, +\infty)$ into \mathbb{R}^d endowed with the metric ϱ defined by

$$\varrho(w, v) = \sum_{N=1}^{\infty} 2^{-N} \sup_{t \leq N} |w(t) - v(t)| \wedge 1 \quad (5)$$

for all $w, v \in C([0, +\infty))$. Let $\mathcal{C}([0, +\infty))$ be the σ -algebra of Borel subsets of $C([0, +\infty))$. We notice that $\mathcal{C}([0, +\infty)) = \mathcal{E}([0, +\infty)) \cap C([0, +\infty))$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space and $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be an increasing family of sub- σ -algebras of \mathcal{F} , called a filtration. We suppose that $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions, i.e., is right-continuous and \mathcal{F}_0 contains all \mathcal{F} -sets of \mathbf{P} -measure zero. For a process $X = (X_t)_{t \geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbf{P})$ we write (X, \mathbb{F}) for X being \mathbb{F} -adapted. If Z is a random variable on $(\Omega, \mathcal{F}, \mathbf{P})$ with values in a measurable space (E, \mathcal{E}) , $\mathcal{D}_{\mathbf{P}}(Z)$ will frequently be used as synonymous notation for the distribution \mathbf{P}_Z of Z with respect to \mathbf{P} on (E, \mathcal{E}) .

Let now X^n , $n \in \mathbb{N}$, and X be stochastic processes with trajectories in a metric space S defined on probability spaces $(\Omega^n, \mathcal{F}^n, \mathbf{P}^n)$ and $(\Omega, \mathcal{F}, \mathbf{P})$, respectively. If the sequence $(\mathbf{P}_{X^n}^n)_{n \in \mathbb{N}}$ of distributions of X^n converges weakly to the distribution \mathbf{P}_X of X , so we shall write

$$\lim_{n \rightarrow \infty} \mathcal{D}_{\mathbf{P}^n}(X^n) = \mathcal{D}_{\mathbf{P}}(X).$$

We shall repeatedly make use of the following rule for weak convergence. The proof is the same as in P. Billingsley [4], Theorem 4.2. Let (S, d) be a separable metric space and $\mathcal{B}(S)$ the σ -algebra of Borel subsets. In our situation, either $S = C([0, +\infty))$ and d is the metric ϱ introduced in (5) or $S = C([0, +\infty)) \times C([0, +\infty))$ endowed with the product metric $d = \varrho^2$.

Proposition 2.2 *Let X_k^n and Y^n , X_k , and X be random variables defined on probability spaces $(\Omega^n, \mathcal{F}^n, \mathbf{P}^n)$, $(\Omega_k, \mathcal{F}_k, \mathbf{P}_k)$, and $(\Omega, \mathcal{F}, \mathbf{P})$, respectively, with values in $(S, \mathcal{B}(S))$. Suppose that the following conditions are satisfied:*

- 1) $\lim_{n \rightarrow \infty} \mathcal{D}_{\mathbf{P}^n}(X_k^n) = \mathcal{D}_{\mathbf{P}_k}(X_k)$.
- 2) $\lim_{k \rightarrow \infty} \mathcal{D}_{\mathbf{P}_k}(X_k) = \mathcal{D}_{\mathbf{P}}(X)$.
- 3) $\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P}^n(d(X_k^n, Y^n) \geq \varepsilon) = 0$ for all $\varepsilon > 0$.

Then we have

$$\lim_{n \rightarrow \infty} \mathcal{D}_{\mathbf{P}^n}(Y^n) = \mathcal{D}_{\mathbf{P}}(X).$$

A stochastic process (X, \mathbb{F}) , defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and with trajectories in $E([0, +\infty))$, is called a solution of Eq. (1) with initial state $x_0 \in \mathbb{R}^d$ if there exists a d -dimensional Wiener process $W = (W_t)_{t \geq 0}$ with respect to the filtration \mathbb{F} such that $W_0 = 0$ and

$$X_t = x_0 + \int_0^t B(s, X_s) dW_s \quad \text{on } \{t < \tau_\Delta(X)\} \quad \mathbf{P}\text{-a.s.} \quad (6)$$

for all $t \geq 0$ where $\tau_\Delta(X)$ is the composition of τ_Δ (defined by (2)) and X and is called the explosion time of X .

Solutions of this type are sometimes called *weak* solutions. Let $\tau_m(X)$ be the composition of τ_m and X which is an \mathbb{F} -stopping time. Obviously, Eq. (6) is equivalent to

$$X_{t \wedge \tau_m(X)} = x_0 + \int_0^{t \wedge \tau_m(X)} B(s, X_s) dW_s \quad \mathbf{P}\text{-a.s.}, \quad t \geq 0, \quad m \in \mathbb{N}. \quad (7)$$

We notice that $(\tau_m(X))_{m \in \mathbb{N}}$ is a localizing sequence for the continuous local martingale up to $\tau_\Delta(X)$ given in (6). Therefore, the processes in (7) are bounded martingales with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$.

Let $B : [0, +\infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^{d^2}$ be a Borel measurable matrix function. We define σ by

$$\sigma(t, x) = B(t, x) \circ B^*(t, x), \quad (t, x) \in [0, +\infty) \times \mathbb{R}^d,$$

where A^* denotes the transpose of a matrix A . Clearly, $\sigma(t, x)$ is a symmetric and nonnegative definite matrix. Hence we can find orthogonal matrices $U(t, x)$, which can be chosen measurable in (t, x) , such that

$$\Lambda(t, x) = U^*(t, x) \circ \sigma(t, x) \circ U(t, x), \quad (t, x) \in [0, +\infty) \times \mathbb{R}^d, \quad (8)$$

are of diagonal form with nonnegative diagonal elements $\lambda_i(t, x)$, $i = 1, 2, \dots, d$. Equivalently, σ has the representation

$$\sigma(t, x) = U(t, x) \circ \Lambda(t, x) \circ U^*(t, x), \quad (t, x) \in [0, +\infty) \times \mathbb{R}^d. \quad (9)$$

The following chain of inequalities can easily be verified:

$$\max_{i,j=1,\dots,d} \sigma_{ij} \leq \max_{i=1,\dots,d} \lambda_i \leq \text{trace } \sigma \leq d \max_{i=1,\dots,d} \sigma_{ii} \quad (10)$$

where A_{ij} , $i, j = 1, 2, \dots, d$, denotes the entries of a matrix A . The next lemma is elementary but crucial for later estimates.

Lemma 2.3 *Let*

$$\lambda_i^{(n)} = (\lambda_i \vee \frac{1}{n}) \wedge n, \quad i = 1, 2, \dots, d, \quad n \in \mathbb{N}.$$

We then have the inequalities:

$$\left(\max_{i=1,\dots,d} \lambda_i^{(n)} \right)^d \left(\prod_{i=1}^d \lambda_i^{(n)} \right)^{-1} \leq \max_{i=1,\dots,d} (\lambda_i + 1)^d \left(\prod_{i=1}^d \lambda_i \right)^{-1} \quad (11)$$

$$\left(\prod_{i=1}^d \lambda_i^{(n)} \right)^{-1} \leq 2^d \max_{i=1,\dots,d} (\lambda_i + 1)^d \left(\prod_{i=1}^d \lambda_i \right)^{-1} \quad (12)$$

where $0^{-1} = +\infty$.

Proof. Fixing $(t, x) \in [0, +\infty) \times \mathbb{R}^d$ we may assume that $\lambda_1, \lambda_2, \dots, \lambda_d$ are nonnegative real numbers. If $\lambda_i = 0$ for some $i = 1, 2, \dots, d$ then the inequalities are obvious. Hence we may assume that $\lambda_1, \lambda_2, \dots, \lambda_d$ are strictly positive. First we consider the case that $\lambda_1, \lambda_2, \dots, \lambda_d$ are bounded from below by $\frac{1}{n}$. Fix $k = 1, 2, \dots, d$ and let $M = \{i : \lambda_k^{(n)} \leq \lambda_i^{(n)}\}$. Clearly, if $i \notin M$ then $\lambda_i^{(n)} < \lambda_k^{(n)} \leq n$ and hence $\lambda_i^{(n)} = \lambda_i$. Consequently, setting $m = \text{card } M$ we obtain

$$\begin{aligned} \left(\lambda_k^{(n)} \right)^d \left(\prod_{i=1}^d \lambda_i^{(n)} \right)^{-1} &\leq \left(\lambda_k^{(n)} \right)^{d-m} \prod_{i \in M} \lambda_i^{(n)} \left(\prod_{i=1}^d \lambda_i^{(n)} \right)^{-1} \\ &\leq \lambda_k^{d-m} \left(\prod_{i \notin M} \lambda_i^{(n)} \right)^{-1} \\ &= \lambda_k^{d-m} \left(\prod_{i \notin M} \lambda_i \right)^{-1} \leq \lambda_k^{d-m} \left(\prod_{i \in M} \lambda_i \right) \left(\prod_{i=1}^d \lambda_i \right)^{-1} \\ &\leq \max_{i=1,\dots,d} \lambda_i^d \left(\prod_{i=1}^d \lambda_i \right)^{-1}. \end{aligned}$$

Next let the sequence $\lambda_1, \lambda_2, \dots, \lambda_d$ be arbitrary but strictly positive. Applying the inequality just derived to the new sequence $(\lambda_k \vee \frac{1}{n})_{k=1,2,\dots,d}$ we obtain

$$\begin{aligned} \left(\lambda_k^{(n)}\right)^d \left(\prod_{i=1}^d \lambda_i^{(n)}\right)^{-1} &\leq \max_{i=1,\dots,d} \left(\lambda_i \vee \frac{1}{n}\right)^d \left(\prod_{i=1}^d \left(\lambda_i \vee \frac{1}{n}\right)\right)^{-1} \\ &\leq \max_{i=1,\dots,d} (\lambda_i + 1)^d \left(\prod_{i=1}^d \lambda_i\right)^{-1}. \end{aligned}$$

Since k was chosen arbitrarily, this finishes the proof of (11). For verifying (12), we observe

$$\begin{aligned} \left(\prod_{i=1}^d \lambda_i^{(n)}\right)^{-1} &\leq \prod_{i=1}^d (\lambda_i^{-1} + 1) = \sum_{M \subseteq \{1,2,\dots,d\}} \prod_{i \in M} \lambda_i^{-1} \\ &= \left(\sum_{M \subseteq \{1,2,\dots,d\}} \prod_{i \in M^c} \lambda_i\right) \prod_{i=1}^d \lambda_i^{-1} \\ &\leq 2^d \max_{i=1,\dots,d} (\lambda_i + 1)^d \left(\prod_{i=1}^d \lambda_i\right)^{-1}. \end{aligned}$$

This finishes the proof of Lemma 2.3. \square

Next we state a version of Krylov's estimates for stochastic integrals (also see A. Rozkosz and L. Słomiński [17], [19]) which will be essential for the proof of our main result. Let (X, \mathbb{F}) be a solution of Eq. (1). For all $m \in \mathbb{N}$ we define the ball $U_m = \{x \in \mathbb{R}^d : |x| \leq m\}$ around the origin with radius m .

Lemma 2.4 *Let $f : [0, +\infty) \times \mathbb{R}^d \rightarrow [0, +\infty)$ be a nonnegative measurable function. Then there exists a constant C which depends only on t, m and d such that the following inequality holds:*

$$\mathbf{E} \int_0^{t \wedge \tau_m(X)} f(s, X_s) [\det \sigma(s, X_s)]^{\frac{1}{d+1}} ds \leq C \left(\int_{[0,t] \times U_m} f^{d+1}(s, y) dy ds \right)^{\frac{1}{d+1}}.$$

Proof. Let us first assume that f is nonnegative, bounded and continuous. For $t \geq 0$ and $m \in \mathbb{N}$ we set

$$g(t-s, x) = \begin{cases} f(s, x) & \text{if } s \in [0, t], x \in U_m, \\ 0 & \text{otherwise.} \end{cases}$$

According to Lemma 2.2.7 of N.V. Krylov [12] there is a function z of $\mathbb{R} \times \mathbb{R}^d$ into \mathbb{R} such that the following conditions are satisfied:

- 1) $z(t, x) \leq 0$ for all $(t, x) \in [0, +\infty) \times \mathbb{R}^d$.

2) For all sufficiently small $\varepsilon > 0$ and $(s, x) \in [0, t] \times U_m$, for some constant $N_1 = N_1(d, t, m)$ we have

$$\begin{aligned} & N_1 \left[\det \frac{1}{2} \sigma(s, x) \right]^{\frac{1}{d+1}} g^{(\varepsilon)}(t - s, x) \\ & \leq - \frac{\partial z^{(\varepsilon)}}{dt}(t - s, x) + \sum_{i,j=1}^d \frac{\partial^2 z^{(\varepsilon)}}{\partial x_i \partial x_j}(t - s, x) \frac{1}{2} \sigma_{ij}(s, x) \end{aligned} \quad (13)$$

where $g^{(\varepsilon)}$ and $z^{(\varepsilon)}$ are the convolutions of g and z with a function φ_ε defined by $\varphi_\varepsilon(u, x) = \varepsilon^{-d-1} \varphi(\varepsilon^{-1}(u, x))$ with some nonnegative infinitely often differentiable function φ having compact support and satisfying

$$\int_{[0, +\infty) \times \mathbb{R}^d} \varphi(u, x) du dx = 1.$$

3) For some constant $N_2 = N_2(d, t, m)$

$$|z(t, x)| \leq N_2 \left(\int_{[0, t] \times U_m} g^{d+1}(s, y) dy ds \right)^{\frac{1}{d+1}}. \quad (14)$$

Using (13), an application of the multidimensional Itô formula leads to

$$\begin{aligned} & 2^{-\frac{d}{d+1}} N_1 \int_0^{t \wedge \tau_m(X)} \left[\det \sigma(s, X_s) \right]^{\frac{1}{d+1}} g^{(\varepsilon)}(t - s, X_s) ds \\ & \leq - \int_0^{t \wedge \tau_m(X)} \frac{\partial z^{(\varepsilon)}}{dt}(t - s, X_s) ds + \frac{1}{2} \sum_{i,j=1}^d \int_0^{t \wedge \tau_m(X)} \frac{\partial^2 z^{(\varepsilon)}}{\partial x_i \partial x_j}(t - s, X_s) \sigma_{ij}(s, X_s) ds \\ & = - \int_0^{t \wedge \tau_m(X)} \frac{\partial z^{(\varepsilon)}}{dt}(t - s, X_s) ds + \frac{1}{2} \sum_{i,j=1}^d \int_0^{t \wedge \tau_m(X)} \frac{\partial^2 z^{(\varepsilon)}}{\partial x_i \partial x_j}(t - s, X_s) d\langle X^i, X^j \rangle_s \\ & = z^{(\varepsilon)}(t - t \wedge \tau_m(X), X_{t \wedge \tau_m(X)}) - z^{(\varepsilon)}(t, x) - \sum_{i=1}^d \int_0^{t \wedge \tau_m(X)} \frac{\partial z^{(\varepsilon)}}{\partial x_i}(t - s, X_s) dX_s^i \end{aligned}$$

where $(\langle X^i, X^j \rangle)_{i,j=1,2,\dots,d}$ denotes the covariation matrix of the continuous local martingale (up to $\tau_\Delta(X)$). Taking the expectation in the latter inequality and using (14) we obtain

$$\begin{aligned} & \mathbf{E} \int_0^{t \wedge \tau_m(X)} \left[\det \sigma(s, X_s) \right]^{\frac{1}{d+1}} g^{(\varepsilon)}(t - s, X_s) ds \\ & \leq 2^{\frac{d}{d+1}} N_1^{-1} \left(\mathbf{E} \left[z^{(\varepsilon)}(t - t \wedge \tau_m(X), X_{t \wedge \tau_m(X)}) \right] - z^{(\varepsilon)}(t, x) \right) \end{aligned}$$

$$\begin{aligned}
&\leq 4N_1^{-1} \sup_{s \leq t, x \in U_m} |z^{(\varepsilon)}(t-s, x)| \\
&\leq 4N_1^{-1} N_2 \left(\int_{[0, t] \times U_m} g^{d+1}(u, y) dy du \right)^{\frac{1}{d+1}} \\
&= 4N_1^{-1} N_2 \left(\int_{[0, t] \times U_m} f^{d+1}(u, y) dy du \right)^{\frac{1}{d+1}}.
\end{aligned}$$

Since g is continuous on $[0, t] \times U_m$, $g^{(\varepsilon)}$ converges to g as $\varepsilon \rightarrow 0$ and by Fatou's lemma

$$\begin{aligned}
&\mathbf{E} \int_0^{t \wedge \tau_m(X)} [\det \sigma(s, X_s)]^{\frac{1}{d+1}} f(s, X_s) ds \\
&\leq \liminf_{\varepsilon \rightarrow 0} \mathbf{E} \int_0^{t \wedge \tau_m(X)} [\det \sigma(s, X_s)]^{\frac{1}{d+1}} g^{(\varepsilon)}(t-s, X_s) ds \\
&\leq C \left(\int_{[0, t] \times U_m} f^{d+1}(s, y) dy ds \right)^{\frac{1}{d+1}}.
\end{aligned}$$

Now we obtain the inequality stated in Lemma 2.4 for functions $f = |h|$ where h is an arbitrary bounded continuous function on $[0, t] \times U_m$. Using the monotone class theorem (cf. P.A. Meyer [15], Theorem I.20 and the following remarks) we observe that the inequality remains valid for all bounded measurable h and hence for all nonnegative bounded measurable functions f . Finally, in the general case f can be approximated increasingly by the nonnegative bounded functions $f \wedge n$. This finishes the proof of the lemma. \square

From Lemma 2.4 immediately follows

Lemma 2.5 *Suppose that $\det \sigma(s, y) \neq 0$ for almost all $(s, y) \in [0, t] \times U_m$. For any nonnegative measurable function f , $m \in \mathbb{N}$ and $t > 0$ we then have*

$$\mathbf{E} \int_0^{t \wedge \tau_m(X)} f(s, X_s) ds \leq C \left(\int_{[0, t] \times U_m} f^{d+1}(s, y) [\det \sigma(s, y)]^{-1} dy ds \right)^{\frac{1}{d+1}}$$

where C is a constant as in Lemma 2.4.

3 Existence of Solutions

Let f be a measurable function on $[0, +\infty) \times \mathbb{R}^d$. We will use the notation $f \in L^{\text{loc}}([0, +\infty) \times \mathbb{R}^d)$ if f is locally integrable, i.e., integrable with respect to the Lebesgue measure on every compact subset of $[0, +\infty) \times \mathbb{R}^d$. Obviously, the function $\det \sigma = \det B \circ B^*$ is nonnegative. We define the measure μ on $[0, +\infty) \times \mathbb{R}^d$ by

$$d\mu(s, y) = [\det \sigma(t, y)]^{-1} dy ds \tag{15}$$

where $0^{-1} = +\infty$. Similarly, the notation $f \in L^{\text{loc}}([0, +\infty) \times \mathbb{R}^d, \mu)$ stands for the local integrability of f with respect to the measure μ on $[0, +\infty) \times \mathbb{R}^d$.

We need the following two conditions:

- a) $(\det B \circ B^*)^{-1} \in L^{\text{loc}}([0, +\infty) \times \mathbb{R}^d)$.
- b) $\|B\|^{2(d+1)} \in L^{\text{loc}}([0, +\infty) \times \mathbb{R}^d, \mu)$.

Obviously, we have

$$\|B\|^2 := \sum_{i,j=1}^d B_{ij}^2 = \text{trace } \sigma$$

which we frequently use without mentioning it every time. In dimension $d = 1$, the conditions a) and b) amount to saying that the functions B^{-2} and B^2 are locally integrable in $[0, +\infty) \times \mathbb{R}$. Under these conditions, T. Senf [20] was able to prove the existence of a (possibly, exploding) solution to Eq. (1). The next theorem generalizes this result to the multidimensional case.

Theorem 3.1 *Suppose that the conditions a) and b) are satisfied. Then, for an arbitrary $x_0 \in \mathbb{R}^d$, there exists a solution X of Eq. (1) with $X_0 = x_0$.*

Proof. Let the matrix functions U and Λ be defined as in (8) and (9). As above λ_i , $i = 1, 2, \dots, d$, denote the diagonal elements of Λ . For $n \in \mathbb{N}$ we consider the diagonal matrix function Λ_n with diagonal $\lambda_i^{(n)} = (\lambda_i \vee \frac{1}{n}) \wedge n$, $i = 1, 2, \dots, d$. Now we define

$$B_n = U \circ \Lambda_n^{\frac{1}{2}} \circ U^* \quad \text{and} \quad \sigma^{(n)} = B_n \circ B_n^*.$$

In view of $\|B_n\|^2 = \text{trace } \sigma^{(n)}$ and (10) we get

$$\|B_n\|^2 \leq dn, \quad n \in \mathbb{N}.$$

Furthermore, for every $z \in \mathbb{R}^d$

$$(B_n z, z) = (U \Lambda_n^{\frac{1}{2}} U^* z, z) = (\Lambda_n^{\frac{1}{2}} U^* z, U^* z) \geq n^{-\frac{1}{2}} |U^* z|^2 = n^{-\frac{1}{2}} |z|^2.$$

Therefore, the coefficients B_n satisfy the assumptions of Krylov's theorem (cf. [12], Theorem 2.6.1) and hence there exist probability spaces $(\Omega^n, \mathcal{F}^n, \mathbf{P}^n)$ with filtrations $\mathbb{F}^n = (\mathcal{F}_t^n)_{t \geq 0}$ and processes (X^n, \mathbb{F}^n) and (W^n, \mathbb{F}^n) such that (W^n, \mathbb{F}^n) are Wiener processes and (X^n, W^n) satisfy Eq. (1) with initial value $X_0^n = x_0$.

Now we show that the sequence of processes $(X_{t \wedge \tau_m}^n)_{n \in \mathbb{N}}$, for arbitrary but fixed m , is tight in $C([0, +\infty))$. Due to a theorem of D. Aldous [2], for tightness in the Skorohod space $D([0, +\infty))$, it is sufficient to show that for every sequence (τ^n) of \mathbb{F}^n -stopping times and every sequence (δ_n) of real numbers such that $\delta_n \downarrow 0$ and all $\varepsilon > 0$ it follows

$$\lim_{n \rightarrow \infty} \mathbf{P}^n(|X_{t \wedge (\tau^n + \delta_n) \wedge \tau_m}^n - X_{t \wedge \tau^n \wedge \tau_m}^n| > \varepsilon) = 0. \quad (16)$$

Because the processes $X_{\cdot \wedge \tau_m}^n$ are continuous, from [10], Theorem VI.3.26, then follows that the tightness also holds in $C([0, +\infty))$. We now verify (16): By Tchebychev's inequality and Lemma 2.5, for any $K \geq 1$ we obtain

$$\begin{aligned}
& \mathbf{P}^n(|X_{t \wedge (\tau^n + \delta_n) \wedge \tau_m}^n - X_{t \wedge \tau^n \wedge \tau_m}^n| > \varepsilon) \\
& \leq \varepsilon^{-2} \mathbf{E}^n \int_{t \wedge \tau^n \wedge \tau_m}^{t \wedge (\tau^n + \delta_n) \wedge \tau_m} \text{trace } \sigma^{(n)}(s, X_s^n) ds \\
& \leq K \delta_n \varepsilon^{-2} + \varepsilon^{-2} \mathbf{E}^n \int_0^{t \wedge \tau_m} \mathbf{1}_{\{\text{trace } \sigma^{(n)} > K\}} \text{trace } \sigma^{(n)}(s, X_s^n) ds \\
& \leq K \delta_n \varepsilon^{-2} + \varepsilon^{-2} C \left(\int_{[0, t] \times U_m} \mathbf{1}_{\{\text{trace } \sigma^{(n)} > K\}} (\text{trace } \sigma^{(n)})^{d+1} (\det \sigma^{(n)})^{-1} dy ds \right)^{\frac{1}{d+1}}.
\end{aligned}$$

From (10) we get

$$d^{-1} \text{trace } \sigma^{(n)} \leq \lambda^{(n)} := \max_{i=1, \dots, d} \lambda_i^{(n)}.$$

Clearly, $\det \sigma^{(n)} = \det \Lambda_n$ and using Lemma 2.3 and the obvious inequalities $\lambda_i^{(n)} \leq \lambda_i + 1$, $i = 1, \dots, d$, we observe

$$(\text{trace } \sigma^{(n)})^{d+1} (\det \sigma^{(n)})^{-1} \leq d^{d+1} (\lambda + 1)^{d+1} (\det \Lambda)^{-1}$$

where $\lambda := \max_{i=1, \dots, d} \lambda_i$. Now $\det \Lambda = \det \sigma$ and, again applying the inequalities (10), we obtain

$$(\text{trace } \sigma^{(n)})^{d+1} (\det \sigma^{(n)})^{-1} \leq d^{2(d+1)} \left(\max_{i=1, \dots, d} (\sigma_{ii}) + 1 \right)^{d+1} (\det \sigma)^{-1}.$$

Setting $\gamma = \max_{i=1, \dots, d} \sigma_{ii} + 1$ we arrive at

$$\begin{aligned}
& \mathbf{P}^n(|X_{t \wedge (\tau^n + \delta_n) \wedge \tau_m}^n - X_{t \wedge \tau^n \wedge \tau_m}^n| > \varepsilon) \\
& \leq K \delta_n \varepsilon^{-2} + d^2 \varepsilon^{-2} C \left(\int_{[0, t] \times U_m} \mathbf{1}_{\{\gamma > K d^{-2}\}} \gamma^{d+1} (\det \sigma)^{-1} dy ds \right)^{\frac{1}{d+1}}.
\end{aligned}$$

But in view of conditions a) and b) the function γ^{d+1} is locally integrable with respect to $d\mu = (\det \sigma)^{-1} dy ds$ and, consequently, the right hand side converges to zero for $n \rightarrow \infty$ and then $K \rightarrow \infty$. This shows that $(X_{\cdot \wedge \tau_m}^n)_{n \in \mathbb{N}}$ is tight in $C([0, +\infty))$.

Using the well-known theorem of Yu.V. Prochorov (cf. [4], Theorem 6.1), by the diagonal method we can choose a subsequence (n_k) and, for every $m \in \mathbb{N}$, probability measures $\tilde{\mathbf{R}}^m$ on $C([0, +\infty))$ such that

$$\lim_{k \rightarrow \infty} \mathcal{D}_{\mathbf{P}^{n_k}}(X_{\cdot \wedge \tau_m}^{n_k}) = \tilde{\mathbf{R}}^m.$$

For simplicity we assume that

$$\lim_{n \rightarrow \infty} \mathcal{D}_{\mathbf{P}^n}(X_{\cdot \wedge \tau_m}^n) = \tilde{\mathbf{R}}^m \quad \text{for all } m \in \mathbb{N}. \quad (17)$$

Let us extend $\tilde{\mathbf{R}}^m$ to probability measures \mathbf{R}^m on $(E([0, +\infty)), \mathcal{E}([0, +\infty)))$ by

$$\mathbf{R}^m(A) = \tilde{\mathbf{R}}^m(A \cap C([0, +\infty))), \quad A \in \mathcal{E}([0, +\infty)). \quad (18)$$

We recall the definition of the coordinate mappings $Z = (Z_t)_{t \geq 0}$ on $E([0, +\infty))$ by (3) and denote their restrictions to $C([0, +\infty))$ by $\tilde{Z} = (\tilde{Z}_t)_{t \geq 0}$. Similarly, let $\tilde{\tau}_a$ be the restrictions of the \mathbb{E} -stopping times τ_a defined by (2). We note that $\tilde{\tau}_a$ is lower semicontinuous and $\tilde{\tau}_{a+} := \lim_{b \downarrow a} \tilde{\tau}_b$ is upper semicontinuous. Furthermore, it can easily be seen that $\tilde{\tau}_a = \tilde{\tau}_{a+}$ \mathbf{R}^m -a.s. for all $m \in \mathbb{N}$ and for all $a \geq 0$ except for, possibly, a countable set of numbers a (cf. [23]). Hence there is a sequence $(a_m)_{m \in \mathbb{N}}$ with $a_m \in (m-1, m]$ such that $\tilde{\tau}_{a_m}$ are \mathbf{R}^m -a.s. and also $\tilde{\mathbf{R}}^{m+1}$ -a.s. continuous functions on $C([0, +\infty))$.

To simplify the notation, we write

$$\vartheta_m = \tau_{a_m} \quad \text{and} \quad \tilde{\vartheta}_m = \tilde{\tau}_{a_m}$$

and introduce the continuous processes ${}^m X^n$ and ${}^m Z$ defined on $(\Omega^n, \mathcal{F}^n, \mathbf{P}^n)$ and $(E([0, +\infty)), \mathcal{E}([0, +\infty)))$, respectively, by

$${}^m X_t^n = X_{t \wedge \vartheta_m}^n \quad \text{and} \quad {}^m Z_t = Z_{t \wedge \vartheta_m}, \quad t \geq 0.$$

The restriction of ${}^m Z$ to $C([0, +\infty))$ is denoted by ${}^m \tilde{Z}$. In the sequel, as image space for ${}^m X^n$ and ${}^m Z$ we consider $(C([0, +\infty)), \mathcal{C}([0, +\infty)))$. We define the probability measures $\tilde{\mathbf{Q}}^m$ by

$$\tilde{\mathbf{Q}}^m = \mathcal{D}_{\mathbf{R}^m}({}^m Z) \quad (= \mathcal{D}_{\tilde{\mathbf{R}}^m}({}^m \tilde{Z})),$$

i.e., $\tilde{\mathbf{Q}}^m$ on $(C([0, +\infty)), \mathcal{C}([0, +\infty)))$ is the distribution of the stopped process ${}^m Z$ defined on the probability space $(E([0, +\infty)), \mathcal{E}([0, +\infty)), \mathbf{R}^m)$. The probability measures \mathbf{Q}^m on $(E([0, +\infty)), \mathcal{E}([0, +\infty)))$ are now introduced as the extensions of $\tilde{\mathbf{Q}}^m$ analogously to (18).

From (17) and the continuous mapping theorem (cf. [4], Theorem 5.1) we obtain

$$\lim_{n \rightarrow \infty} \mathcal{D}_{\mathbf{P}^n}({}^m X^n) = \tilde{\mathbf{Q}}^m \quad \text{for all } m \in \mathbb{N} \quad (19)$$

and, for the same reasons,

$$\lim_{n \rightarrow \infty} \mathcal{D}_{\mathbf{P}^n}({}^m X^n) = \mathcal{D}_{\mathbf{R}^{m+1}}({}^m Z) \quad \text{for all } m \in \mathbb{N}$$

because $(a_m)_{m \in \mathbb{N}}$ is chosen such that $\vartheta_m = \tau_{a_m}$ is continuous \mathbf{R}^m -a.s. and \mathbf{R}^{m+1} -a.s. This yields the equality

$$\mathcal{D}_{\mathbf{R}^{m+1}}({}^m Z) = \tilde{\mathbf{Q}}^m \quad \text{for all } m \in \mathbb{N}. \quad (20)$$

We now state the following

Lemma 3.2 *For all $m \in \mathbb{N}$ we have*

- 1) $\mathcal{D}_{\mathbf{Q}^m}({}^mZ) = \tilde{\mathbf{Q}}^m$.
- 2) $\mathcal{D}_{\mathbf{Q}^{m+1}}({}^mZ) = \mathcal{D}_{\mathbf{Q}^m}({}^mZ)$.
- 3) $\mathbf{Q}^{m+1}(A) = \mathbf{Q}^m(A)$ for all $A \in \mathcal{E}_{\vartheta_m}$.

Proof. 1) follows from the identity ${}^mZ \circ {}^mZ = {}^mZ$ and the definition of $\tilde{\mathbf{Q}}^m$ and \mathbf{Q}^m . Using ${}^mZ \circ {}^{m+1}Z = {}^mZ$ and relation (20) we observe

$$\mathcal{D}_{\mathbf{Q}^{m+1}}({}^mZ) = \mathcal{D}_{\mathbf{R}^{m+1}}({}^mZ) = \tilde{\mathbf{Q}}^m$$

which implies 2) in view of 1). Finally, 3) is a simple consequence of 2) and the property ${}^mZ^{-1}(A) = A$ for every $A \in \mathcal{C}_{\tilde{\vartheta}_m} := \mathcal{E}_{\vartheta_m} \cap \mathcal{C}([0, +\infty))$. This proves Lemma 3.2. \square

In view of Lemma 3.2 and Proposition 2.1 there exists a unique probability measure \mathbf{Q} on $(E([0, +\infty)), \mathcal{E}([0, +\infty)))$ such that

$$\mathbf{Q}(A) = \mathbf{Q}^m(A) \quad \text{for all } A \in \mathcal{E}_{\vartheta_m}, \quad m \in \mathbb{N}.$$

The definition of \mathbf{Q} , the representation (4) and Lemma 3.2 yield

$$\mathcal{D}_{\mathbf{Q}}({}^mZ) = \mathcal{D}_{\mathbf{Q}^m}({}^mZ) = \tilde{\mathbf{Q}}^m.$$

Hence statement (19) can be rewritten as

$$\lim_{n \rightarrow \infty} \mathcal{D}_{\mathbf{P}^n}({}^mX^n) = \mathcal{D}_{\mathbf{Q}}({}^mZ) \quad \text{for all } m \in \mathbb{N}. \quad (21)$$

Thus we constructed processes mZ on $(E([0, +\infty)), \mathcal{E}([0, +\infty)), \mathbf{Q})$ to which the sequence $({}^mX^n)_{n \in \mathbb{N}}$ converges weakly. Our aim is to show that the process Z defined on the completed probability space $(E([0, +\infty)), \mathcal{E}^{\mathbf{Q}}([0, +\infty)), \mathbf{Q})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, $\mathcal{F}_t = \mathcal{E}_{t+}^{\mathbf{Q}}$, $t \geq 0$, where the superscript \mathbf{Q} means completion in $\mathcal{E}^{\mathbf{Q}}([0, +\infty))$, is the desired solution of Eq. (1).

For this we notice that in view of [10], Proposition IX.1.10, $({}^mZ, \mathbb{F}^{mZ})$ is a continuous martingale where \mathbb{F}^{mZ} denotes the filtration generated by mZ . From this easily follows that $({}^mZ, \mathbb{F})$ is also a continuous martingale. This yields that (Z, \mathbb{F}) is a continuous local martingale up to the explosion time τ_{Δ} . In order to prove that (Z, \mathbb{F}) is a solution of Eq. (1) it is now sufficient to show that for the covariation matrix of $Z = (Z^1, Z^2, \dots, Z^d)$ we have

$$\langle Z^i, Z^j \rangle_t = \int_0^t \sigma_{ij}(s, Z_s) ds \quad \text{on } \{t < \tau_{\Delta}\} \quad \mathbf{Q}\text{-a.s.}$$

or, equivalently,

$$\langle {}^mZ^i, {}^mZ^j \rangle_t = \int_0^t \sigma_{ij}(s, Z_s) ds \quad \text{on } \{t < \vartheta_m\} \quad \mathbf{Q}\text{-a.s.} \quad \text{for all } m \in \mathbb{N} \quad (22)$$

and for every $i, j = 1, 2, \dots, d$ (cf. [9], Theorem 7.1'). The result will be proven if we are able to verify

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathcal{D}_{\mathbf{P}^n}(\langle {}^m X^{ni}, {}^m X^{nj} \rangle, \int_0^{\cdot \wedge \vartheta_m(X^n)} \sigma_{ij}^{(n)}(s, X_s^n) ds) \\ &= \mathcal{D}_{\mathbf{Q}}(\langle {}^m Z^i, {}^m Z^j \rangle, \int_0^{\cdot \wedge \vartheta_m} \sigma_{ij}(s, Z_s) ds) \end{aligned} \quad (23)$$

for all $m \in \mathbb{N}$ and $i, j = 1, 2, \dots, d$. Indeed, regarding that X^n is a solution to (1) for B_n , an application of the continuous mapping theorem (cf. [4], Theorem 5.1) to the metric ϱ then implies

$$\varrho(\langle {}^m Z^i, {}^m Z^j \rangle, \int_0^{\cdot \wedge \vartheta_m} \sigma_{ij}(s, Z_s) ds) = 0$$

for all $m \in \mathbb{N}$ and $i, j = 1, 2, \dots, d$ and, consequently, (22) is satisfied. For proving (23), we first fix $p \in \mathbb{N}$ and show

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathcal{D}_{\mathbf{P}^n}(\langle {}^m X^{ni}, {}^m X^{nj} \rangle, \int_0^{\cdot \wedge \vartheta_m(X^n)} \sigma_{ij}^{(p)}(s, X_s^n) ds) \\ &= \mathcal{D}_{\mathbf{Q}}(\langle {}^m Z^i, {}^m Z^j \rangle, \int_0^{\cdot \wedge \vartheta_m} \sigma_{ij}^{(p)}(s, Z_s) ds) \end{aligned} \quad (24)$$

for all $m \in \mathbb{N}$ and $i, j = 1, 2, \dots, d$. Since $\sigma_{ij}^{(p)}$ is bounded by p and hence $(\sigma_{ij}^{(p)})^{d+1}$ is locally integrable we can choose a sequence $(f_{ij}^{(k)})_{k \in \mathbb{N}}$ of continuous functions uniformly bounded by p such that

$$\lim_{k \rightarrow \infty} \int_{[0, N] \times U_m} |f_{ij}^{(k)} - \sigma_{ij}^{(p)}|^{d+1} dy ds = 0$$

for all $m, N \in \mathbb{N}$. By [10], Corollary VI.6.6, and the continuous mapping theorem (cf. [4], Theorem 5.1) we now obtain

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathcal{D}_{\mathbf{P}^n}(\langle {}^m X^{ni}, {}^m X^{nj} \rangle, \int_0^{\cdot \wedge \vartheta_m(X^n)} f_{ij}^{(k)}(s, X_s^n) ds) \\ &= \mathcal{D}_{\mathbf{Q}}(\langle {}^m Z^i, {}^m Z^j \rangle, \int_0^{\cdot \wedge \vartheta_m} f_{ij}^{(k)}(s, Z_s) ds). \end{aligned} \quad (25)$$

Next we extend Lemma 2.4 to the process Z .

Lemma 3.3 For any nonnegative measurable function f , $m \in \mathbb{N}$ and $t > 0$ we have

$$\mathbf{E}_{\mathbf{Q}} \int_0^{t \wedge \vartheta_m} f(s, Z_s) ds \leq C \left(\int_{[0,t] \times U_m} f^{d+1}(s, y) [\det \sigma(s, y)]^{-1} dy ds \right)^{\frac{1}{d+1}}.$$

Proof. We can assume that f is continuous (cf. proof of Lemma 2.4). Using (21), the continuous mapping theorem, Fatou's Lemma (cf. [4], Theorem 5.1 and Theorem 5.3) and Lemma 2.5, we conclude

$$\begin{aligned} \mathbf{E}_{\mathbf{Q}} \int_0^{t \wedge \vartheta_m} f(s, Z_s) ds &\leq \liminf_{n \rightarrow \infty} \mathbf{E}^n \int_0^{t \wedge \vartheta_m} f(s, X_s^n) ds \\ &\leq C \liminf_{n \rightarrow \infty} \left(\int_{[0,t] \times U_m} f^{d+1} [\det \sigma^{(n)}]^{-1} dy ds \right)^{\frac{1}{d+1}} \\ &= C \left(\int_{[0,t] \times U_m} f^{d+1} [\det \sigma]^{-1} dy ds \right)^{\frac{1}{d+1}} \end{aligned}$$

where the last equality follows from $\lim_{n \rightarrow \infty} \det \sigma^{(n)} = \det \sigma$ a.e. and the uniform integrability of $\det \sigma^{(n)}$ on $[0, t] \times U_m$ which is derived from (12) in Lemma 2.3, inequalities (10) and assumptions a) and b) of the theorem. \square

Using Lemma 3.3 we estimate

$$\begin{aligned} &\mathbf{Q} \left(\sup_{0 \leq t \leq N} \left| \int_0^{t \wedge \vartheta_m} f_{ij}^{(k)}(s, Z_s) ds - \int_0^{t \wedge \vartheta_m} \sigma_{ij}^{(p)}(s, Z_s) ds \right| > \varepsilon \right) \\ &\leq \varepsilon^{-1} \mathbf{E}_{\mathbf{Q}} \int_0^{N \wedge \vartheta_m} |f_{ij}^{(k)}(s, Z_s) - \sigma_{ij}^{(p)}(s, Z_s)| ds \\ &\leq \varepsilon^{-1} C \left(\int_{[0,N] \times U_m} |f_{ij}^{(k)} - \sigma_{ij}^{(p)}|^{d+1} [\det \sigma]^{-1} dy ds \right)^{\frac{1}{d+1}}. \end{aligned}$$

Since $|f_{ij}^{(k)} - \sigma_{ij}^{(p)}|^{d+1}$ is bounded by $(2p)^{d+1}$ condition a) yields that the right hand side converges to zero as $k \rightarrow \infty$ and hence

$$\lim_{k \rightarrow \infty} \mathcal{D}_{\mathbf{Q}}(\langle {}^m Z^i, {}^m Z^j \rangle, \int_0^{\cdot \wedge \vartheta_m} f_{ij}^{(k)}(s, Z_s) ds) = \mathcal{D}_{\mathbf{Q}}(\langle {}^m Z^i, {}^m Z^j \rangle, \int_0^{\cdot \wedge \vartheta_m} \sigma_{ij}^{(p)}(s, Z_s) ds). \quad (26)$$

In the next step we estimate

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \mathbf{P}^n \left(\sup_{0 \leq t \leq N} \left| \int_0^{t \wedge \vartheta_m(X^n)} f_{ij}^{(k)}(s, X_s^n) ds - \int_0^{t \wedge \vartheta_m(X^n)} \sigma_{ij}^{(p)}(s, X_s^n) ds \right| > \varepsilon \right) \\ &\leq \varepsilon^{-1} \limsup_{n \rightarrow \infty} \mathbf{E}^n \int_0^{N \wedge \vartheta_m(X^n)} |f_{ij}^{(k)}(s, X_s^n) - \sigma_{ij}^{(p)}(s, X_s^n)| ds \\ &\leq \varepsilon^{-1} C \limsup_{n \rightarrow \infty} \left(\int_{[0,N] \times U_m} |f_{ij}^{(k)} - \sigma_{ij}^{(p)}|^{d+1} [\det \sigma^{(n)}]^{-1} dy ds \right)^{\frac{1}{d+1}}. \end{aligned}$$

As in the proof of Lemma 3.3, $(\det \sigma^{(n)})^{-1}$ is uniformly integrable over $[0, N] \times U_m$. Since $\lim_{k \rightarrow \infty} f_{ij}^{(k)} = \sigma_{ij}^{(p)}$ a.e. and $f_{ij}^{(k)}$ is bounded by p as well as $\sigma_{ij}^{(p)}$ the right hand side converges to zero as $k \rightarrow \infty$. Using the abbreviations

$${}^m I_{ij}^{nk}(t) = \int_0^{t \wedge \vartheta_m(X^n)} f_{ij}^{(k)}(s, X_s^n) ds \quad \text{and} \quad {}^m I_{ij}^n(t) = \int_0^{t \wedge \vartheta_m(X^n)} \sigma_{ij}^{(p)}(s, X_s^n) ds,$$

this gives

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P}^n(\varrho^2(\langle \langle {}^m X^{ni}, {}^m X^{nj} \rangle, {}^m I_{ij}^{nk} \rangle; \langle \langle {}^m X^{ni}, {}^m X^{nj} \rangle, {}^m I_{ij}^n \rangle) > \varepsilon) = 0 \quad (27)$$

for all $\varepsilon > 0$.

Applying Proposition 2.2, from relations (25), (26) and (27) we obtain the weak convergence (24).

For the final step $p \rightarrow \infty$ we first observe that

$$\lim_{p \rightarrow \infty} \mathcal{D}_{\mathbf{Q}}(\langle \langle {}^m Z^i, {}^m Z^j \rangle, \int_0^{\cdot \wedge \vartheta_m} \sigma_{ij}^{(p)}(s, Z_s) ds \rangle) = \mathcal{D}_{\mathbf{Q}}(\langle \langle {}^m Z^i, {}^m Z^j \rangle, \int_0^{\cdot \wedge \vartheta_m} \sigma_{ij}(s, Z_s) ds \rangle) \quad (28)$$

because in view of Lemma 3.3

$$\begin{aligned} & \mathbf{Q}(\sup_{0 \leq t \leq N} | \int_0^{t \wedge \vartheta_m} \sigma_{ij}^{(p)}(s, Z_s) ds - \int_0^{t \wedge \vartheta_m} \sigma_{ij}(s, Z_s) ds | > \varepsilon) \\ & \leq \varepsilon^{-1} \mathbf{E}_{\mathbf{Q}} \int_0^{N \wedge \vartheta_m} |\sigma_{ij}^{(p)}(s, Z_s) - \sigma_{ij}(s, Z_s)| ds \\ & \leq \varepsilon^{-1} C \left(\int_{[0, N] \times U_m} |\sigma_{ij}^{(p)} - \sigma_{ij}|^{d+1} [\det \sigma]^{-1} dy ds \right)^{\frac{1}{d+1}}. \end{aligned}$$

By majorized convergence the right term converges to zero as $p \rightarrow \infty$ since by the definition of $\sigma^{(p)}$ it follows $\lim_{p \rightarrow \infty} |\sigma_{ij}^{(p)} - \sigma_{ij}| = 0$ and by (10)

$$\begin{aligned} |\sigma_{ij}^{(p)} - \sigma_{ij}|^{d+1} [\det \sigma]^{-1} & \leq \max_{k=1,2,\dots,d} |\lambda_k^{(p)} - \lambda_k|^{d+1} [\det \sigma]^{-1} \\ & \leq \max_{k=1,2,\dots,d} (\lambda_k + 1)^{d+1} [\det \sigma]^{-1} \\ & \leq (\text{trace } \sigma + 1)^{d+1} [\det \sigma]^{-1} \end{aligned}$$

which is integrable over $[0, N] \times U_m$.

Furthermore, by Lemma 2.5 and inequalities (10) we have

$$\begin{aligned} \delta_p & := \limsup_{n \rightarrow \infty} \mathbf{P}^n(\sup_{0 \leq t \leq N} | \int_0^{t \wedge \vartheta_m(X^n)} \sigma_{ij}^{(n)}(s, X_s^n) ds - \int_0^{t \wedge \vartheta_m(X^n)} \sigma_{ij}^{(p)}(s, X_s^n) ds | > \varepsilon) \\ & \leq \varepsilon^{-1} \limsup_{n \rightarrow \infty} \mathbf{E}_n \int_0^{N \wedge \vartheta_m(X^n)} |\sigma_{ij}^{(n)}(s, X_s^n) - \sigma_{ij}^{(p)}(s, X_s^n)| ds \end{aligned}$$

$$\begin{aligned}
&\leq \varepsilon^{-1} C \limsup_{n \rightarrow \infty} \left(\int_{[0, N] \times U_m} |\sigma_{ij}^{(n)} - \sigma_{ij}^{(p)}|^{d+1} [\det \sigma^{(n)}]^{-1} dy ds \right)^{\frac{1}{d+1}} \\
&\leq \varepsilon^{-1} C \limsup_{n \rightarrow \infty} \left(\int_{[0, N] \times U_m} \max_{k=1,2,\dots,d} |\lambda_k^{(n)} - \lambda_k^{(p)}|^{d+1} [\det \sigma^{(n)}]^{-1} dy ds \right)^{\frac{1}{d+1}} \\
&\leq \varepsilon^{-1} C \limsup_{n \rightarrow \infty} \left(\int_{[0, N] \times U_m} \max_{k=1,2,\dots,d} (\lambda_k^{(n)})^{d+1} 1_{\{\lambda_k > p\}} [\det \sigma^{(n)}]^{-1} dy ds \right. \\
&\quad \left. + \int_{[0, N] \times U_m} \max_{k=1,2,\dots,d} p^{-(d+1)} 1_{\{\lambda_k \leq \frac{1}{p}\}} [\det \sigma^{(n)}]^{-1} dy ds \right)^{\frac{1}{d+1}} \\
&\leq \varepsilon^{-1} C \left(\int_{[0, N] \times U_m} \max_{k=1,2,\dots,d} (\lambda_k + 1)^{d+1} 1_{\{\lambda_k > p\}} [\det \sigma]^{-1} dy ds \right. \\
&\quad \left. + p^{-(d+1)} 2^d \int_{[0, N] \times U_m} \max_{k=1,2,\dots,d} (\lambda_k + 1)^{d+1} [\det \sigma]^{-1} dy ds \right)^{\frac{1}{d+1}}
\end{aligned}$$

where the last inequality follows from $\lambda_k^{(n)} \leq \lambda_k + 1$ and Lemma 2.3. In view of the conditions a) and b) and the inequalities (10), the function

$$\max_{k=1,2,\dots,d} (\lambda_k + 1)^{d+1} [\det \sigma]^{-1}$$

being integrable over $[0, N] \times U_m$, we observe that the right hand side converges to zero as $p \rightarrow \infty$. This yields $\lim_{p \rightarrow \infty} \delta_p = 0$. Consequently, with the abbreviations

$$m_{ij}^n(t) = \int_0^{t \wedge \vartheta_m(X^n)} \sigma_{ij}^{(n)}(s, X_s^n) ds \quad \text{and} \quad m_{ij}^{np}(t) = \int_0^{t \wedge \vartheta_m(X^n)} \sigma_{ij}^{(p)}(s, X_s^n) ds,$$

we conclude

$$\lim_{p \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P}^n(\varrho^2(\langle mX^{ni}, mX^{nj} \rangle, m_{ij}^n); (\langle mX^{ni}, mX^{nj} \rangle, m_{ij}^{np})) > \varepsilon) = 0 \quad (29)$$

for all $\varepsilon > 0$. Finally, using Proposition 2.2, from (24), (28) and (29) we obtain (23). The proof of Theorem 3.1 is completed. \square

Theorem 3.1 can be extended in the following way. We introduce the sets

$$N_B = \{(t, x) \in [0, +\infty) \times \mathbb{R}^d : B(s, x) = 0 \text{ for almost all } s \geq t\}$$

and

$$E_B = \{(t, x) \in [0, +\infty) \times \mathbb{R}^d : \mu(S_\delta(t, x)) = \infty \text{ for all } \delta > 0\}$$

where $S_\delta(t, x)$ denotes the ball with center (t, x) and radius δ in $[0, +\infty) \times \mathbb{R}^d$. We recall the definition of the measure μ by (15). Clearly, $E_B^c := [0, +\infty) \times$

$\mathbb{R}^d \setminus E_B$ is an open subset of $[0, +\infty) \times \mathbb{R}^d$ and μ is a locally finite measure on E_B^c , i.e., $\mu(K) < +\infty$ for every compact subset of E_B^c . We now state the following conditions:

- A) $E_B \subseteq N_B$.
- B) $\|B\|^{2(d+1)} \in L^{\text{loc}}(E_B^c, \mu)$.

The condition $f \in L^{\text{loc}}(E_B^c, \mu)$ means that f is integrable with respect to μ over every compact subset K of E_B^c (and not of $[0, +\infty) \times \mathbb{R}^d$). Now we state

Theorem 3.4 *Suppose that the conditions A) and B) are satisfied. Then, for any $x_0 \in \mathbb{R}^d$, there exists a solution X of Eq. (1) with $X_0 = x_0$.*

Proof. The proof is very close to the proof of Theorem 3.1 and therefore we only give a brief sketch. Let $(D_m)_{m \in \mathbb{N}}$ be an increasing sequence of compact subsets of E_B^c such that $E_B^c = \bigcup_{m \in \mathbb{N}} D_m^0$ where D_m^0 denotes the interior of D_m . Without loss of generality we can assume $D_m \subseteq [0, m] \times U_m$ for every $m \in \mathbb{N}$. By $\tau_{D_m^0}$ we denote the \mathbb{E} -stopping time

$$\tau_{D_m^0}(w) = \inf\{t \geq 0 : w(t) \notin D_m^0\}, \quad w \in E([0, +\infty)).$$

Then Lemma 2.4 and Lemma 2.5 remain valid if we replace $t \wedge \tau_m(X)$ by $\tau_{D_m^0}(X)$ on the left hand side and the set $[0, t] \times U_m$ by D_m on the right hand side. Indeed, in the left integral with upper bound $\tau_{D_m^0}(X)$ we first replace f by $f 1_{D_m}$ and then the upper bound $\tau_{D_m^0}(X)$ by the larger $m \wedge \tau_m(X)$ and now apply Lemma 2.4 to obtain the result.

Now we define B_n as in the proof of Theorem 3.1 and let X^n be solutions of Eq. (1) for the coefficients B_n defined on probability spaces $(\Omega^n, \mathcal{F}^n, \mathbf{P}^n)$. Exactly as in the proof of Theorem 3.1 we can verify that, for every $m \in \mathbb{N}$, the sequence of stopped processes $(X_{\cdot \wedge \tau_{D_m^0}}^n)_{n \in \mathbb{N}}$ is tight and, consequently, we can construct a subsequence (n_k) and probability measures $\tilde{\mathbf{R}}^m$ on $(C([0, +\infty)), \mathcal{C}([0, +\infty)))$ such that

$$\lim_{k \rightarrow \infty} \mathcal{D}_{\mathbf{P}^{n_k}}(X_{\cdot \wedge \tau_{D_m^0}}^{n_k}) = \tilde{\mathbf{R}}^m$$

for all $m \in \mathbb{N}$. For simplicity we assume

$$\lim_{n \rightarrow \infty} \mathcal{D}_{\mathbf{P}^n}(X_{\cdot \wedge \tau_{D_m^0}}^n) = \tilde{\mathbf{R}}^m \quad \text{for all } m \in \mathbb{N}.$$

Let \mathbf{R}^m be the extensions on $(E([0, +\infty)), \mathcal{E}([0, +\infty))$ of the probability measures $\tilde{\mathbf{R}}^m$ (cf. (18)). Let now G_m be open subsets of D_m^0 with $E_B^c = \bigcup_{m \in \mathbb{N}} G_m$ such that the restriction to $C([0, +\infty))$ of the first exit time τ_{G_m} of Z from G_m is continuous \mathbf{R}^m -a.s. and \mathbf{R}^{m+1} -a.s. We then define

$$\tau_\infty = \lim_{m \rightarrow \infty} \tau_{G_m}$$

and notice that τ_∞ is just the first entry time into the closed subset E_B of $[0, +\infty) \times \mathbb{R}^d$. By statements analogous to Proposition 2.1 and Lemma 3.2 there

exists a unique probability measure \mathbf{Q} on $\mathcal{E}_{\tau_\infty}$ which extends the restrictions of \mathbf{R}^m to $\mathcal{E}_{\tau_{G_m}}$ for $m \in \mathbb{N}$. In the same way as in the proof of Theorem 3.1 it is now possible to prove that (Z, \mathbb{F}) is a continuous local martingale up to τ_∞ and

$$\langle Z^i, Z^j \rangle_t = \int_0^t \sigma_{ij}(s, Z_s) ds \quad \text{on } \{t < \tau_\infty\} \quad \mathbf{Q}\text{-a.s.}$$

for all $i, j = 1, 2, \dots, d$. This, however, amounts to saying that Z is a solution to Eq. (1) up to the stopping time τ_∞ . Finally, setting $Z_t^\infty = Z_{t \wedge \tau_\infty}$, condition A) ensures that Z^∞ is, indeed, a solution to Eq. (1) on the probability space $(E([0, +\infty)), \mathcal{E}^{\mathbf{Q}}([0, +\infty)), \mathbf{Q})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, $\mathcal{F}_t = \mathcal{E}_{t+}^{\mathbf{Q}}$, $t \geq 0$. \square

If the initial value lies in E_B^c , i.e., $(0, x_0) \in E_B^c$, instead of condition A) it suffices to require

$$\mathbf{A}') \quad \partial E_B \subseteq N_B$$

where ∂E_B denotes the boundary of E_B .

Remark 3.5 (i) Theorem 3.1 and 3.4 ensure the existence of, possibly, only exploding solutions (X, \mathbb{F}) of Eq. (1). If we are looking for nonexploding solutions this should be understood as a first step. If we have found any solution, in a second step one could ask for nonexplosion conditions. It seems that a general answer to this problem is hardly possible, unless the linear growth condition is exploited. Nonexplosion and explosion strongly depend on the dimension and on the concrete structure of the diffusion matrix. In the one-dimensional case, however, explosion never occurs if B is time-independent. In the general, time-dependent case for nonexplosion of solutions it is sufficient to know (under the conditions of Theorem 3.1 or 3.4) that $\sup_{0 \leq t \leq N} |B(t, x)|$ is finite at least on a set of strictly positive Lebesgue measure, for every $N \geq 1$ (cf. T. Senf [20]). This illustrates that the linear growth condition as condition for nonexplosion of solutions is not the best possible.

(ii) A.S. Kosciuk [11] (see also S. Albeverio et al. [1]) gave the following existence condition for degenerate but bounded diffusion matrices B , $\sigma = B \circ B^*$: Let

$$\Lambda = \{(t, x) \in [0, +\infty) \times \mathbb{R}^d : \exists (t_n, x_n) \rightarrow (t, x) \quad \text{and} \quad \det \sigma(t_n, x_n) \rightarrow 0\}$$

and assume that the restriction of B to Λ is continuous and, moreover, if B is not continuous at (t, x) then $\det \sigma$ is bounded away from zero on $U \setminus \Lambda$ for every sufficiently small neighbourhood U of (t, x) .

(iii) If B is of at most linear growth, the following remarkable existence condition (specialized to our situation) was found by A. Rozkosz and L. Słomiński [19]:

$$\bar{V} \subseteq N_B \tag{30}$$

where

$$\begin{aligned}
V &= [(E_B \setminus E_B^1) \cap D] \cup [E_b^1 \cap D^1], \\
E_B^1 &= \{(t, x) \in [0, +\infty) \times \mathbb{R}^d : \mu(S_\delta(t, x) \setminus E_B) = +\infty \text{ for all } \delta > 0\}, \\
D &= \{(t, x) \in E_B : B(t, \cdot) \text{ restricted to } E_B \text{ is discontinuous at } x\}, \\
D^1 &= \{(t, x) \in [0, +\infty) \times \mathbb{R}^d : B(t, \cdot) \text{ is discontinuous at } x\}.
\end{aligned}$$

For example, if $B(t, \cdot)$ is continuous for all $t \geq 0$, then (30) is satisfied without further conditions. It also includes the above stated condition of A.S. Kosciuk [11]. We believe that Theorem 3.4 can also be proven using the condition (30) instead of A). However, what concerns condition A) we did not strive for full generality because it seems that in the multidimensional case condition A) as well as condition (30) are far from a necessary condition for existence of solutions. These conditions allow only relatively small degeneracies of B in comparison with what should be possible. Undoubtedly, it is of physical relevance that solutions of Eq. (1) are living on lower dimensional manifolds, possibly varying in time. It seems that it is a difficult task to give a complete description of the degenerate case. The following trivial examples should illustrate the situation.

Example 3.6 For simplicity, let $d = 2$ and $B(t, x) = B(x)$, $(t, x) \in [0, +\infty) \times \mathbb{R}^2$. As above, we set $\sigma = B \circ B^*$. In the following two simple examples we always have $\det \sigma \equiv 0$ and, in particular, $E_B = [0, +\infty) \times \mathbb{R}^2$. Additionally, we can ensure that $N_B = \emptyset$ and that B is nowhere continuous, hence $D = [0, +\infty) \times \mathbb{R}^2$ and, consequently, $V = [0, +\infty) \times \mathbb{R}^2$. (Here we used the notations introduced in Remark 3.5 (iii).) But nevertheless a solution of Eq. (1) does exist for all initial states $x_0 \in \mathbb{R}^2$, without assuming any continuity properties.

(i) Let $B_{11} = B_{12} = B_{21} \equiv 0$ and assume that $b := B_{22}$ has the property that $b^{-2}(x^1, x^2)$ is locally integrable with respect to x^2 for every $x^1 \in \mathbb{R}$. Then, for every $x_0 = (x_0^1, x_0^2) \in \mathbb{R}^2$, there exists a solution (X, \mathbb{F}) to Eq. (1) with initial state x_0 . Indeed, we can find a two-dimensional Brownian motion $((W^1, W^2), \mathbb{F})$ and a process (X^2, \mathbb{F}) on some probability space such that

$$X_t^2 = x_0^2 + \int_0^t b(x_0^1, X_s^2) dW_s^2, \quad t \geq 0$$

(cf. [6]). Now we set $X_t^1 \equiv x_0^1$, $t \geq 0$. Obviously, (X, \mathbb{F}) with $X = (X^1, X^2)$ is then a solution to Eq. (1) for the diffusion matrix B .

(ii) In the second example we choose $B_{11} = B_{21} =: b$ and $B_{12} = B_{22} \equiv 0$ and we assume that the function $h_c(x) = b(x, c + x)$, $x \in \mathbb{R}$ has the property that h_c^{-2} is locally integrable for every $c \in \mathbb{R}$. Let $x_0 = (x_0^1, x_0^2) \in \mathbb{R}^2$. Then there exists a two-dimensional Brownian motion $((W^1, W^2), \mathbb{F})$ and a process (X^1, \mathbb{F}) on some probability space such that for $c = x_0^2 - x_0^1$ we have

$$X_t^1 = x_0^1 + \int_0^t b(X_s^1, c + X_s^1) dW_s^1, \quad t \geq 0$$

(cf. [6]). We set $X_t^2 = c + X_t^1$, $t \geq 0$, and $X = (X^1, X^2)$. Then, obviously, (X, \mathbb{F}) is a solution to Eq. (1) for the diffusion matrix B .

4 The Homogeneous Case

In this section we will briefly discuss the homogeneous case, i.e., the case where the matrix function B does not depend on $t \in [0, +\infty)$ and thus is a function $B : \mathbb{R}^d \rightarrow \mathbb{R}^{d^2}$. A matrix function of this type we shall call homogeneous. In this situation the sufficient conditions on B for the existence of a solution can be improved. For this the key is the following improvement of Lemma 2.4.

Lemma 4.1 *Let B be a homogeneous matrix function and (X, \mathbb{F}) be a solution of Eq. (1). Suppose that $f : \mathbb{R}^d \rightarrow [0, +\infty)$ is a nonnegative measurable function. Then there exists a constant C which depends only on t , m and d such that the following inequality holds:*

$$\mathbf{E} \int_0^{t \wedge \tau_m(X)} f(X_s) [\det \sigma(X_s)]^{\frac{1}{d}} ds \leq C \left(\int_{[0,t] \times U_m} f^d(y) dy ds \right)^{\frac{1}{d}}.$$

For the proof cf. [18], Lemma 1.

Analogously to Section 3, we introduce the sets

$$\bar{N}_B = \{x \in \mathbb{R}^d : B(x) = 0\}$$

and

$$\bar{E}_B = \{x \in \mathbb{R}^d : \bar{\mu}(S_\delta(x)) = \infty \text{ for all } \delta > 0\}$$

where $S_\delta(x)$ denotes the ball with center x and radius δ in \mathbb{R}^d . Here the measure $\bar{\mu}$ is defined by

$$\bar{\mu}(A) = \int_A (\det \sigma(y))^{-1} dy, \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Clearly, \bar{E}_B^c is an open subset of \mathbb{R}^d and $\bar{\mu}$ is a locally finite measure on \bar{E}_B^c . Now the conditions A) and B) are changed as follows:

$$\begin{aligned} \bar{\mathbf{A}}) \quad & \bar{E}_B \subseteq \bar{N}_B. \\ \bar{\mathbf{B}}) \quad & \|B\|^{2d} \in L^{\text{loc}}(\bar{E}_B^c, \bar{\mu}). \end{aligned}$$

In conclusion, we state the homogeneous version of Theorem 3.4. If, additionally, B satisfies the growth condition $\|B \circ B^*(x)\| \leq K(1 + |x|^2)$, $x \in \mathbb{R}^d$, this result was proven by A. Rozkosz and L. Słomiński [18].

Theorem 4.2 *Suppose that for the homogeneous matrix function B the conditions \bar{A}) and \bar{B}) are satisfied. Then, for every $x_0 \in \mathbb{R}^d$, there exists a solution X of Eq. (1) with $X_0 = x_0$.*

The proof is similar to the proofs of Theorems 3.1 and 3.4 and will therefore be omitted.

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