On Multidimensional SDEs with Locally Integrable Coefficients

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third revision: August 30, 2005

Abstract

We consider multidimensional stochastic equations

$$X_t = x_0 + \int_0^t B(s, X_s)\, dW_s + \int_0^t A(s, X_s) ds$$

where x_0 is an arbitrary initial value, W is a d-dimensional Wiener process and $B : [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}^{d^2}$, $A : [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ are measurable diffusion and drift coefficients, respectively. Our main result states sufficient conditions for the existence of (possibly, exploding) weak solutions. These conditions are some local integrability conditions of coefficients B and A. From one side, they extend the conditions from [3] where the corresponding SDEs without drift were considered. On the other hand, our results generalize the existence theorems for one-dimernsional SDEs with drift studied in [4]. We also discuss the time-independent case.

Key Words: Multidimensional stochastic differential equations, locally integrable coefficients, Krylov's estimates, Wiener process, weak convergence

MR Subject Classification 60H10, 60J60, 60J65, 60G44

1 Introduction

In this note we consider a stochastic equation of the form

$$X_t = x_0 + \int_0^t B(s, X_s)\, dW_s + \int_0^t A(s, X_s) ds, \quad t \geq 0, \tag{1.1}$$

where the coefficients $B : [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}^{d^2}$, $A : [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ are Borel measurable matrix- and vector-valued functions with $d \geq 1$, respectively, W is a d-dimensional Wiener process and $x_0 \in \mathbb{R}^d$ is an arbitrary initial vector.

It is well-known that if the coefficients A and B satisfy the assumption of at most linear growth, that is if there exists a constant $K_1 > 0$ such that

$$|A(t, x)| + \|B(t, x)\| \leq K_1(1 + |x|) \quad \text{for all} \quad t \geq 0, x \in \mathbb{R}^d, \tag{1.2}$$

then the solution of the equation, if it exists, is nonexploding, i.e., it exists in $\mathbb{R}^d$ for all $t \geq 0$ (cf. [6], Theorem 6.4.2). Here $|\cdot|$ denotes the Euclidean norm of a vector in $\mathbb{R}^d$ and

$$\|B(t, x)\|^2 = \sum_{i,j=1}^{d} B_{ij}^2(t, x).$$

In the general case the solution exists only in the sense that it may explode, i.e., on a finite time interval it may leave every compact subset of $\mathbb{R}^d$. In this paper we study solutions of equation (1.1) in this more general context.

The purpose of this article is to prove the existence results for the equation (1.1) in the weak sense. This equation is also called the Itô equation because K. Itô was the first who considered it [7]. He proved that the equation (1.1) has a solution if the coefficients A and B satisfy the condition (1.2) and there exists a constant $K_2 > 0$ such that

$$|A(t, x) - A(t, y)| + \|B(t, x) - B(t, y)\| \leq K_2|x - y| \quad \text{for all} \quad t \geq 0, x, y \in \mathbb{R}^d. \tag{1.3}$$

The coefficients A and B satisfying the condition (1.3) are said to be globally Lipshitz continuous.

A.V. Skorokhod [23] proved later the existence of a solution for coefficients not satisfying the Lipshitz condition. More precisely, he replaced the condition of Lipshitz continuity of the coefficients by the condition of their usual continuity in the space variable. At the same time, A.V. Skorokhod proposed a different concept of the solution than that used by K. Itô: A.V. Skorokhod was looking for a solution of the equation existing on a not a priori fixed probability space while following the K. Itô's idea one had to find a solution on a given probability space with a given process W. Since then one started distinguishing two concepts of a solution for (1.1): strong solutions (in the sense of K. Itô) and weak solutions (in the sense of A.V. Skorokhod). The names "strong" and "weak" reflect the fact that any strong solution is also a solution in the weak sense but not vice versa.

The conditions of A.V. Skorokhod were essentially weakened by N.V. Krylov [10] who proved the existence of weak solutions of the stochastic equation (1.1) for discontinuous and measurable coefficients using his well-known estimates for stochastic integrals of diffusion processes. He assumed the coefficients A and B to satisfy the following conditions: there exist constants $C > 0$ and $0 < c_1 < c_2$ not depending on $(t, x) \in [0, +\infty) \times \mathbb{R}^d$ such that for all $x, z \in \mathbb{R}^d$, and $t \geq 0$

$$|A(t, x)| \leq C, \qquad c_1 |z|^2 \leq \langle\langle B(t, x)z, z\rangle\rangle, \quad \|B(t, x)\|^2 \leq c_2,$$

where $\langle\langle\cdot, \cdot\rangle\rangle$ denotes the Euclidean scalar product.

The case of one-dimensional homogeneous equations (i.e., equations with time-independent coefficients) was treated in detail by H.J. Engelbert and W. Schmidt (cf. [4], [5]). To formulate some of their results, define the sets

$$\mathcal{N} = \{x \in \mathbb{R} : B(x) = 0\} \quad \text{and}$$

$$\mathcal{M} = \{x \in \mathbb{R} : \int\limits_{U(x)} B^{-2}(y)dy = \infty \quad \text{for any open neighbourhood} \quad U(x) \text{ of } x\},$$

where $B^{-2} := 1/B^2$. It was proven that, for any initial value $x_0 \in \mathbb{R}$, the equation (1.1) without drift ($A = 0$) has a solution if and only if $\mathcal{M} \subseteq \mathcal{N}$. In particular, it was shown that the local integrability of B^{-2} is necessary and sufficient for the existence of nontrivial (not equal identically to a constant) solutions with an arbitrary initial value $x_0 \in \mathbb{R}$. They obtained also various sufficient conditions for the existence of solutions of the corresponding time-independent equation with drift. For example, it was proved in [4] that if $B(x) \neq 0$ for all $x \in \mathbb{R}$ and if the function $(1 + |A|)B^{-2}$ is locally integrable, then the homogeneous equation (1.1) has a (possibly, exploding) solution for any initial value $x_0 \in \mathbb{R}$.

A more general case of equation (1.1) with time-dependent coefficient B and $A = 0$ but still with one-dimensional state space was investigated by V.P. Kurenok [12], P. Raupach [17], A. Rozkosz and L. Słomiński [18] and T. Senf [21]. For example, T. Senf [21] was able to prove the existence of a solution for every initial value $x_0 \in \mathbb{R}$ under the following two conditions:

1) B^2 is locally integrable;

2) B^{-2} is locally integrable.

Moreover, it was proven that the solution does not explode if only, for every $N \geq 1$, there exists a nonnegative function $\bar{B}_N$ finite on a set of positive Lebesgue measure such that $B^2(t, x) \leq \bar{B}_N(x)$ for every $t \in [0, N]$ and $x \in \mathbb{R}$. The conditions found in [18] were very similar to those in [21]. In [12] the author assumed the continuity of the function $B^{-2}(t, x)$ in the variable t along with a condition of local integrability of B^{-2}. P. Raupach generalized the result from [21] (respectively, that from [18]) replacing the condition 2) by a weaker condition

2′) $\mathcal{M} \subseteq \mathcal{N}$, where the sets $\mathcal{M}$ and $\mathcal{N}$ in case of the function $B(t,x)$ are defined similarly as in case of the function $B(x)$ above.

Let g be a measurable function in $[0,+\infty)\times \mathbb{R}^d$. We write $g \in L^{\mathrm{loc}}([0,+\infty)\times \mathbb{R}^d)$ if g is locally integrable, i.e., integrable with respect to the Lebesgue measure on every compact subset of $[0,+\infty)\times \mathbb{R}^d$. Let $\sigma = \det B\cdot B^*$ and define the measure μ on $[0,+\infty)\times \mathbb{R}^d$ by

$$d\mu(s,y) = [\det \sigma(t,y)]^{-1}\, dy\, ds \tag{1.4}$$

where $0^{-1} = +\infty$ and B^* is the transpose of the matrix B. Similarly, the notation $g \in L^{\mathrm{loc}}([0,+\infty)\times \mathbb{R}^d, \mu)$ stands for the local integrability of g with respect to the measure μ on $[0,+\infty)\times \mathbb{R}^d$.

In [3] it was proved the existence of weak solutions for multidimensional stochastic equation (1.1) without drift under the following two conditions

$\mathbf{a_1}$) $(\det B\cdot B^*)^{-1} \in L^{\mathrm{loc}}([0,+\infty)\times \mathbb{R}^d)$,

$\mathbf{a_2}$) $\|B\|^{2(d+1)} \in L^{\mathrm{loc}}([0,+\infty)\times \mathbb{R}^d, \mu)$.

The results of this paper generalize the results in [3] for SDEs with drift when the conditions $\mathbf{a_1}$), $\mathbf{a_2}$), and

b) $|A|^{d+1}(\|B\|^{2d}+1) \in L^{\mathrm{loc}}([0,+\infty)\times \mathbb{R}^d, \mu)$

are satisfied.

Another far-reaching generalization was given by A. Rozkosz and L. Słomiński [19], [20] for multidimensional stochastic equations with time-independent and also with time-dependent coefficients A and B satisfying, additionally, the usual linear growth condition (1.2). We refer for the detailed formulation of the results to [19], [20] and only notice that one principal difference between the conditions in [19], [20] and our conditions is that we do not require at most linear growth of the coefficients A and B.

By discussing the results about SDEs with unbounded drift, we cannot leave without mentioning the investigations of N.I. Portenko [16] who proved the existence of (non-exploding) solutions for the equation (1.1) with diffusion coefficient B being uniformly continuous, bounded and nondegenerate, and the drift coefficient $A(t,x)$ being globally Lebesgue integrable in the space variable x of order $p > d+2$ on every interval $[0,T], T > 0$. As an essential tool for proving his results, he used his own estimates (similar to Krylov's estimates) for stochastic integrals of solutions of SDEs with integrable drift. The assumption of global Lebesgue integrability led Portenko to obtain non-exploding solutions while we require the local Lebesgue integrability of coefficients which garantees the existence in more general sense as described above (exploding solutions).

In the time-dependent or multidimensional cases, the main tools remain the Krylov's estimates. We also use here an appropriate variant of Krylov's estimates for processes X satisfying the equation (1.1). The case of time-independent equations is discussed as well.

2 Preliminaries

Let $\bar{\mathbb{R}}^d = \mathbb{R}^d \cup \{\Delta\}$ be the one-point compactification of $\mathbb{R}^d$. By $(\bar{\mathbb{R}}^d, \mathcal{B}(\bar{\mathbb{R}}^d))$ we denote the measurable space generated by the σ-algebra $\mathcal{B}(\bar{\mathbb{R}}^d)$ of Borel subsets of $\bar{\mathbb{R}}^d$. For any function $w : [0, +\infty) \to \bar{\mathbb{R}}^d$ we set

$$\tau_\Delta(w) = \inf\{t \geq 0 : w(t) = \Delta\} \tag{2.1}$$

and call $\tau_\Delta(w)$ the explosion time of the trajectory w. Let $E([0, +\infty), \bar{\mathbb{R}}^d)$ be the set of all right-continuous functions $w : [0, +\infty) \to \bar{\mathbb{R}}^d$ such that w is continuous on $[0, \tau_\Delta(w))$ and $w(t) = \Delta$ whenever $t \geq \tau_\Delta(w)$. For every $t \geq 0$ we define the coordinate mappings $Z_t : E([0, +\infty), \bar{\mathbb{R}}^d) \to \bar{\mathbb{R}}^d$ by

$$Z_t(w) = w(t), \quad w \in E([0, +\infty), \bar{\mathbb{R}}^d), \tag{2.2}$$

and introduce the σ-algebras

$$\mathcal{E}([0, +\infty), \bar{\mathbb{R}}^d) = \sigma(Z_t, t \geq 0), \quad \mathcal{E}_t = \sigma(Z_s, s \leq t), \quad t \geq 0,$$

and the filtration $\mathbb{E} = (\mathcal{E}_t)_{t \geq 0}$. Obviously, τ_Δ is $\mathbb{E}$-stopping time.

Now, for any right-continuous function $w : [0, +\infty) \to \bar{\mathbb{R}}^d$ and any $a > 0$, let $\tau_a(w)$ be a family of $\mathbb{E}$-stopping times such that $\tau_a(w) < \tau_{a'}(w) < \tau_\Delta(w)$ whenever $\tau_\Delta(w) < +\infty$ and $a < a'$. We additionally assume that

$$\tau_a(w) \uparrow \tau_\Delta(w) \text{ as } a \to \infty. \tag{2.3}$$

For example, the family $\tau_a(w)$ of $\mathbb{E}$-stopping times defined by

$$\tau_a(w) := \inf\{t \geq 0 : |w(t)| \geq a\}$$

is an increasing sequence in the sense described above and satisfies the condition (2.3).

We remark further that for every $\mathbb{E}$-stopping time τ the σ-algebra $\mathcal{E}_\tau$ associated with τ can be described as

$$\mathcal{E}_\tau = \sigma(Z_{t \wedge \tau}, t \geq 0) = (Z^\tau)^{-1}(\mathcal{E}([0, +\infty), \bar{\mathbb{R}}^d)), \tag{2.4}$$

where $Z^\tau : E([0, +\infty), \bar{\mathbb{R}}^d) \to E([0, +\infty), \bar{\mathbb{R}}^d)$ is defined by $Z^\tau(w) = w(\cdot \wedge \tau)$ for all $w \in E([0, +\infty), \bar{\mathbb{R}}^d)$ (cf. [22], Theorem I.6).

Now let $(a_m)_{m \in \mathbb{N}}$ be an increasing sequence of positive real numbers converging to infinity as $m \to \infty$. Using standard arguments (cf. T. Senf [21]) one can show that, for every $m \in \mathbb{N}$, $(E([0, +\infty), \bar{\mathbb{R}}^d), \mathcal{E}_{\tau_{a_m}})$ is a standard Borel space (cf. K.R. Parthasarathy [15]). Furthermore, one sees that the σ-algebras $\mathcal{E}_{\tau_{a_m}}$ are increasing, $\mathcal{E}([0, +\infty), \bar{\mathbb{R}}^d) = \sigma(\bigcup_{m \in \mathbb{N}} \mathcal{E}_{\tau_{a_m}})$ in view of the assumtion (2.3), and if $(A_m)_{m \in \mathbb{N}}$ is a decreasing sequence of atoms A_m from $\mathcal{E}_{\tau_{a_m}}$ then $\bigcap_{m \in \mathbb{N}} A_m \neq \emptyset$. Theorem V.4.1 of K.R. Parthasarathy [15] now implies

Proposition 2.1 *Let* $(\mathbf{Q}^m)_{m\in\mathbb{N}}$ *be a consistent family of probability measures* $\mathbf{Q}^m$ *on* $\mathcal{E}_{\tau_{a_m}}$. *Then there exists a unique probability measure* $\mathbf{Q}$ *on* $\mathcal{E}([0,+\infty),\bar{\mathbb{R}}^d)$ *which is an extension of the family* $(\mathbf{Q}^m)_{m\in\mathbb{N}}$.

Let $C([0,+\infty),\mathbb{R}^d) \subset E([0,+\infty),\bar{\mathbb{R}}^d)$ be the space of continuous functions w of $[0,+\infty)$ into $\mathbb{R}^d$ endowed with the metric ρ defined by

$$\rho(w,v) = \sum_{N=1}^{\infty} 2^{-N}\Big(\sup_{t\le N}|w(t)-v(t)| \wedge 1\Big) \tag{2.5}$$

for all $w, v \in C([0,+\infty),\mathbb{R}^d)$. We denote by $\mathcal{C}([0,+\infty),\mathbb{R}^d)$ the σ-algebra of Borel subsets of $C([0,+\infty),\mathbb{R}^d)$ and notice that $\mathcal{C}([0,+\infty),\mathbb{R}^d) = \mathcal{E}([0,+\infty),\bar{\mathbb{R}}^d) \cap C([0,+\infty),\mathbb{R}^d)$.

Let $(\Omega,\mathcal{F},\mathbf{P})$ be a complete probability space and $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$ be a filtration of $\mathcal{F}$. We suppose that $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$ satisfies the usual conditions, i.e., it is right-continuous and $\mathcal{F}_0$ contains all $\mathcal{F}$-sets of $\mathbf{P}$-measure zero. For a process $X = (X_t)_{t\ge 0}$ defined on $(\Omega,\mathcal{F},\mathbf{P})$ we write $(X,\mathbb{F})$ for X being $\mathbb{F}$-adapted. If ξ is a random variable on $(\Omega,\mathcal{F},\mathbf{P})$ with values in a measurable space $(E,\mathcal{E})$, $\mathcal{D}_{\mathbf{P}}(\xi)$ will frequently be used as synonymous notation for the distribution $\mathbf{P}_\xi$ of ξ with respect to $\mathbf{P}$ on $(E,\mathcal{E})$.

Let now X^n, $n\in\mathbb{N}$, and X be stochastic processes with trajectories in a metric space S defined on probability spaces $(\Omega^n,\mathcal{F}^n,\mathbf{P}^n)$ and $(\Omega,\mathcal{F},\mathbf{P})$, respectively. If the sequence $(\mathbf{P}^n_{X^n})_{n\in\mathbb{N}}$ of distributions of X^n converges weakly to the distribution $\mathbf{P}_X$ of X, so we shall write

$$\lim_{n\to\infty} \mathcal{D}_{\mathbf{P}^n}(X^n) = \mathcal{D}_{\mathbf{P}}(X).$$

We shall repeatedly make use of the following rule for weak convergence. The proof is the same as the proof of Theorem 4.2 in P. Billingsley [2]. Let (S,d) be a separable metric space and $\mathcal{B}(S)$ be the σ-algebra of Borel subsets. In this article, $S = C([0,+\infty),\mathbb{R}^d)$ or $S = C([0,+\infty),\mathbb{R}^d) \times C([0,+\infty),\mathbb{R}^d)$ with the metric $d=\rho$ or the product metric $d=\rho^2$ where ρ is introduced in (2.5).

Proposition 2.2 *Let* X^n_k *and* Y^n, X_k, *and* X *be random variables defined on probability spaces* $(\Omega^n,\mathcal{F}^n,\mathbf{P}^n)$, $(\Omega_k,\mathcal{F}_k,\mathbf{P}_k)$, *and* $(\Omega,\mathcal{F},\mathbf{P})$, *respectively, with values in* $(S,\mathcal{B}(S))$. *Suppose that the following conditions are satisfied:*

1) $\lim_{n\to\infty} \mathcal{D}_{\mathbf{P}^n}(X^n_k) = \mathcal{D}_{\mathbf{P}_k}(X_k)$.

2) $\lim_{k\to\infty} \mathcal{D}_{\mathbf{P}_k}(X_k) = \mathcal{D}_{\mathbf{P}}(X)$.

3) $\lim_{k\to\infty} \limsup_{n\to\infty} \mathbf{P}^n(d(X^n_k, Y^n) \ge \varepsilon) = 0$ *for all* $\varepsilon > 0$.

Then we have

$$\lim_{n\to\infty} \mathcal{D}_{\mathbf{P}^n}(Y^n) = \mathcal{D}_{\mathbf{P}}(X).$$

In the sequel we shall use the following norm $\|\cdot\|$ of the matrix B defined as

$$\|B\|^2 := \sum_{i,j=1}^{d} B_{ij}^2 = \text{trace } \sigma.$$

By definition, $\sigma(t,x)$ is a symmetric and nonnegative definite matrix. Hence we can find orthogonal matrices $U(t,x)$, which can be chosen measurable in (t,x), such that

$$\Lambda(t,x) = U^*(t,x)\cdot\sigma(t,x)\cdot U(t,x), \quad (t,x)\in[0,+\infty)\times \mathbb{R}^d, \tag{2.6}$$

are of diagonal form with nonnegative diagonal elements $\lambda_i(t,x)$, $i = 1,2,\cdots,d$. Equivalently, σ has the representation

$$\sigma(t,x) = U(t,x)\cdot\Lambda(t,x)\cdot U^*(t,x), \quad (t,x)\in[0,+\infty)\times \mathbb{R}^d. \tag{2.7}$$

The following chain of inequalities can easily be verified:

$$\max_{i,j=1,\cdots,d}\sigma_{ij} \le \max_{i=1,\cdots,d}\lambda_i \le \text{trace } \sigma \le d \max_{i=1,\cdots,d}\sigma_{ii} \tag{2.8}$$

where $\sigma_{ij}, i,j = 1,2,\cdots,d$, denote the entries of the matrix σ. The next lemma, which is proven in [3] (Lemma 2.4), will be used for later estimates.

Lemma 2.3 *Let*

$$\lambda_i^n = (\lambda_i \vee \frac{1}{n})\wedge n, \quad i = 1,2,\cdots,d, \quad n\in\mathbb{N}.$$

We then have the inequalities:

$$\left(\max_{i=1,\cdots,d}\lambda_i^n\right)^d \left(\prod_{i=1}^d \lambda_i^n\right)^{-1} \le \max_{i=1,\cdots,d}(\lambda_i+1)^d \left(\prod_{i=1}^d \lambda_i\right)^{-1},$$

$$\left(\prod_{i=1}^d \lambda_i^n\right)^{-1} \le 2^d \max_{i=1,\cdots,d}(\lambda_i+1)^d \left(\prod_{i=1}^d \lambda_i\right)^{-1},$$

where $0^{-1} = +\infty$.

A stochastic process $(X,\mathbb{F})$, defined on a probability space $(\Omega,\mathcal{F},\mathbf{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$ and with trajectories in $E([0,+\infty),\bar{\mathbb{R}}^d)$, is called a *solution* of the equation (1.1) with initial value $x_0\in\mathbb{R}^d$ if there exists a d-dimensional Wiener process $W = (W_t)_{t\ge0}$ with respect to the filtration $\mathbb{F}$ such that $W_0 = 0$ and

$$X_t = x_0 + \int_0^t B(s,X_s)\,dW_s + \int_0^t A(s,X_s)ds \quad \text{on} \quad \{t<\tau_\Delta(X)\} \quad \mathbf{P}\text{-a.s.} \tag{2.9}$$

for all $t\ge 0$. Here $\tau_\Delta(X)$, called the *explosion time* of X, is the composition of τ_Δ (defined by (2.1)) and X. Solutions of this type are called *weak* solutions.

Let $(X, \mathbb{F})$ be a solution of equation (2.9) and, for any $m \in \mathbb{N}$, define

$$\tau_m(X) := \tau_m^1(X) \wedge \tau_m^2(X) \wedge \tau_m^3(X), \tag{2.10}$$

where

$$\tau_m^1(X) = \inf\{t \geq 0 : |X_t| \geq m\},$$

$$\tau_m^2(X) = \inf\{t \geq 0 : \int_0^t \text{trace}\, \sigma(s, X_s) ds \geq m\},$$

$$\tau_m^3(X) = \inf\{t \geq 0 : \int_0^t |A(s, X_s)| ds \geq m\}.$$

It can be easily verified that the sequence $\Big(\tau_m(X)\Big), m = 1, 2, \dots$ has the following properties:

1) $\Big(\tau_m(X)\Big)_{m \in \mathbb{N}}$ is a sequence of $\mathbb{F}$ - stopping times;
2) $\tau_m(X) \uparrow \tau_\triangle(X)$ as $m \to \infty$.

Obviously, the equation (2.9) is equivalent to

$$X_{t \wedge \tau_m(X)} = x_0 + \int_0^{t \wedge \tau_m(X)} B(s, X_s)\, dW_s + \int_0^{t \wedge \tau_m(X)} A(s, X_s) ds \quad \mathbf{P}\text{-a.s.}, \tag{2.11}$$

where $t \geq 0, m \in \mathbb{N}$. We notice that $(\tau_m(X))_{m \in \mathbb{N}}$ is a localizing sequence for the continuous semimartingale up to $\tau_\Delta(X)$ given in (2.9). Therefore, the processes in (2.11) are bounded semimartingales with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$.

Next we state a version of Krylov's estimates for stochastic integrals which will be essential for the proof of our main result. As usual, for all $m \in \mathbb{N}$ the set $U_m = \{x \in \mathbb{R}^d : |x| \leq m\}$ determines the ball around the origin with radius m.

Lemma 2.4 *Suppose X is a solution of SDE (1.1) and $f : [0, +\infty) \times \mathbb{R}^d \to [0, +\infty)$ is a nonnegative measurable function. Then there exists a constant C which depends on t, m, and d only such that the following inequality holds:*

$$\mathbf{E}\left[\int_0^{t \wedge \tau_m(X)} f(s, X_s)\ [\det \sigma(s, X_s)]^{\frac{1}{d+1}}\ ds \right] \leq C \Big(\int_{[0,t] \times U_m} f^{d+1}(s, y)\, dy\, ds \Big)^{\frac{1}{d+1}}.$$

Proof. Assume first that f is a nonnegative, bounded, and continuous function and let $t \geq 0$ and $m \in \mathbb{N}$. We set

$$g(s, x) = \begin{cases} f(s, x) & \text{if} \quad s \in [0, t],\ x \in U_m, \\ 0 & \text{otherwise} \end{cases}$$

and take $\varphi \in C_0^\infty([0,\infty)\times \mathbb{R}^d)$ such that

$$\int\limits_{[0,+\infty)\times\mathbb{R}^d} \varphi(u,x)\,du\,dx = 1,$$

where $C_0^\infty([0,\infty)\times\mathbb{R}^d)$ is the set of all infinitely many times differentiable functions with a compact support defined on $[0,\infty)\times\mathbb{R}^d$ and values in $\mathbb{R}$. Furthemore, for any $\varepsilon > 0$ let $g^{(\varepsilon)}(t,x)$ be the convolution of the function $g(t,x)$ with the function φ_ε defined as $\varphi_\varepsilon(u,x) = \varepsilon^{-d-1}\,\varphi(\varepsilon^{-1}(u,x))$. Clearly, $g^{(\varepsilon)}(t,x)\in C_0^\infty([0,\infty)\times\mathbb{R}^d)$.

According to Lemma 5.1 of N.V. Krylov [11] (see also Lemma 1.1 there), there is a function $z^{(\varepsilon)}(s,x)\geq 0$ defined on $[0,t]\times U_m$ such that for all $s\in[0,t]$ and $x\in U_m$ we have:

$$\frac{\partial z^{(\varepsilon)}}{\partial t}(s,x) + \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2 z^{(\varepsilon)}}{\partial x_i \partial x_j}\sigma_{ij}(s,x) \leq$$

$$z^{(\varepsilon)}(s,x)\text{trace}\ \sigma(s,x) - [\det\frac{1}{2}\sigma]^{\frac{1}{d+1}}(s,x)\,g^{(\varepsilon)}(s,x), \tag{2.12}$$

$$|\frac{\partial z^{(\varepsilon)}}{\partial x}(s,x)| \leq z^{(\varepsilon)}(s,x), \tag{2.13}$$

$$z^{(\varepsilon)}(t,x) \leq N\Big(\int\limits_{[0,t]\times U_m}(g^{(\varepsilon)})^{d+1}(s,y)\,dy\,ds\Big)^{\frac{1}{d+1}} := N\,||\,g^{(\varepsilon)}\,||_{m,t,d+1} \tag{2.14}$$

where the constant N depends on d,t and m only.

Applying Ito formula to $z^{(\varepsilon)}(s,X_s)$ leads to

$$\mathbf{E}\left[z^{(\varepsilon)}(t\wedge\tau_m(X), X_{t\wedge\tau_m(X)})\right] - z^{(\varepsilon)}(0,X_0) =$$

$$\mathbf{E}\left[\int\limits_0^{t\wedge\tau_m(X)}\Big(\frac{\partial z^{(\varepsilon)}}{\partial t}(s,X_s) + \frac{1}{2}\sum_{i,j=1}^{d}\sigma_{ij}z^{(\varepsilon)}_{x_i x_j}(s,X_s) + \sum_{i=1}^{d}A_i z^{(\varepsilon)}_{x_i}(s,X_s)\Big)ds\right]. \tag{2.15}$$

Let $s\in[0,t\wedge\tau_m(X)]$. Then using (2.12), (2.13), and the relations

$$A_i \leq \max_i A_i \leq |A|,$$

$$\sum_{i=1}^{d}|\frac{\partial z^{(\varepsilon)}}{\partial x_i}| \leq d\cdot\max_i|\frac{\partial z^{(\varepsilon)}}{\partial x_i}| \leq d|\frac{\partial z^{(\varepsilon)}}{\partial x}|,$$

we obtain

$$\mathbf{E}\left[z^{(\varepsilon)}(t\wedge\tau_m(X), X_{t\wedge\tau_m(X)})\right] - z^{(\varepsilon)}(0,X_0) \leq$$

$$\mathbf{E}\left[\int\limits_0^{t\wedge\tau_m(X)}\Big(z^{(\varepsilon)}(s,X_s)\text{trace}\ \sigma(s,X_s) + d|A(s,X_s)|z^{(\varepsilon)}(s,X_s)\Big)ds\right]$$

$$-\mathbf{E}\left[\int_0^{t\wedge\tau_m(X)} [\det \frac{1}{2}\sigma]^{\frac{1}{d+1}}(s,X_s)g^{(\varepsilon)}(s,X_s)ds\right].$$

Using the nonnegativity of $z^{(\varepsilon)}$ and the relation (2.14), we have

$$-z^{(\varepsilon)}(0,X_0) \le$$

$$\mathbf{E}\left[\int_0^{t\wedge\tau_m(X)} \Big(z^{(\varepsilon)}(s,X_s)\text{trace }\sigma(s,X_s) + d|A(s,X_s)|z^{(\varepsilon)}(s,X_s)\Big)ds\right]$$

$$-\mathbf{E}\left[\int_0^{t\wedge\tau_m(X)} [\det \frac{1}{2}\sigma]^{\frac{1}{d+1}}(s,X_s)g^{(\varepsilon)}(s,X_s)ds\right] \le$$

$$\mathbf{E}\left[\int_0^{t\wedge\tau_m(X)} N\Big(\|g^{(\varepsilon)}\|_{m,t,d+1}\text{trace }\sigma(s,X_s) + d|A(s,X_s)|\|g^{(\varepsilon)}\|_{m,t,d+1}\Big)ds\right]$$

$$-\mathbf{E}\left[\int_0^{t\wedge\tau_m(X)} [\det \frac{1}{2}\sigma]^{\frac{1}{d+1}}(s,X_s)g^{(\varepsilon)}(s,X_s)ds\right]$$

or

$$z^{(\varepsilon)}(0,X_0) \ge$$

$$\mathbf{E}\left[\int_0^{t\wedge\tau_m(X)} [\det \frac{1}{2}\sigma]^{\frac{1}{d+1}}(s,X_s)g^{(\varepsilon)}(s,X_s)ds\right]$$

$$-\mathbf{E}\left[\int_0^{t\wedge\tau_m(X)} N\Big(\|g^{(\varepsilon)}\|_{m,t,d+1}\text{trace }\sigma(s,X_s) + d|A(s,X_s)|\|g^{(\varepsilon)}\|_{m,t,d+1}\Big)ds\right].$$

Then

$$2^{\frac{1}{d+1}}N\left(1+\mathbf{E}\left[\int_0^{t\wedge\tau_m(X)} \text{trace }\sigma(s,X_s)ds\right]+\mathbf{E}\left[\int_0^{t\wedge\tau_m(X)} d|A(s,X_s)|ds\right]\right)\|g^{(\varepsilon)}\|_{m,t,d+1} \ge$$

$$\mathbf{E}\left[\int_0^{t\wedge\tau_m(X)} [\det \sigma]^{\frac{1}{d+1}}(s,X_s)g^{(\varepsilon)}(s,X_s)ds\right].$$

As the result, it follows then from the definition of the sequence $\tau_m(X)$ that

$$\mathbf{E}\left[\int_0^{t\wedge\tau_m(X)} [\det \sigma]^{\frac{1}{d+1}}(s,X_s)g^{(\varepsilon)}(s,X_s)ds\right] \le$$

$$2^{\frac{1}{d+1}} N\Big(1 + 2mtd\Big) \|g^{(\varepsilon)}\|_{m,t,d+1} = C\|g^{(\varepsilon)}\|_{m,t,d+1}.$$

Since g is continuous on $[0,t] \times U_m$, $g^{(\varepsilon)}$ converges to g as $\varepsilon \to 0$ and by Fatou's lemma

$$\mathbf{E}\left[\int_0^{t\wedge\tau_m(X)} [\det\sigma(s,X_s)]^{\frac{1}{d+1}} f(s,X_s)ds\right]$$

$$\leq \liminf_{\varepsilon\to 0} \mathbf{E}\left[\int_0^{t\wedge\tau_m(X)} [\det\sigma(s,X_s)]^{\frac{1}{d+1}} g^{(\varepsilon)}(s,X_s)ds\right] \leq C\Big(\int_{[0,t]\times U_m} f^{d+1}(s,y)dyds\Big)^{\frac{1}{d+1}}.$$

Now we obtain the inequality stated in Lemma 2.4 for functions $f = |h|$ where h is an arbitrary bounded continuous function on $[0,t]\times U_m$. Using the monotone class theorem (cf. P.A. Meyer [14], Theorem I.20 and the following remarks) we observe that the inequality remains valid for all bounded measurable h and hence for all nonnegative bounded measurable functions f. Finally, in the general case f can be approximated increasingly by the nonnegative bounded functions $f \wedge n$. This finishes the proof of the lemma. □

From Lemma 2.4 immediately follows

Corollary 2.5 *Suppose that* $\det\sigma(s,y) \neq 0$ *for almost all* $(s,y) \in [0,t] \times U_m$. *For any nonnegative measurable function* f, $m \in \mathbb{N}$, *and* $t > 0$ *we then have*

$$\mathbf{E}\left[\int_0^{t\wedge\tau_m(X)} f(s,X_s)\,ds\right] \leq C\Big(\int_{[0,t]\times U_m} f^{d+1}(s,y)\ [\det\sigma(s,y)]^{-1}\ dy\,ds\Big)^{\frac{1}{d+1}}$$

where C *is a constant as in Lemma 2.4.*

3 Existence of Solutions

The next theorem generalizes the main existence result in [3] to the case of SDEs with diffusion and drift coefficients.

Theorem 3.1 *Suppose that the conditions* $\mathbf{a_1}), \mathbf{a_2})$ *and* $\mathbf{b})$ *are satisfied. Then, for an arbitrary* $x_0 \in \mathbb{R}^d$, *there exists a solution* X *of the equation (1.1) with* $X_0 = x_0$.

Proof. Let the matrix functions U and Λ be defined as in (2.6) and (2.7). As above λ_i, $i = 1,2,\cdots,d$, denote the diagonal elements of Λ. For $n \in \mathbb{N}$ we consider the diagonal matrix function Λ_n with diagonal entries $\lambda_i^n = (\lambda_i \vee \frac{1}{n}) \wedge n$, $i = 1,2,\cdots,d$, and define

$$B^n = \sqrt{2}U \cdot \Lambda_n^{\frac{1}{2}} \cdot U^*, \qquad \sigma^n = \frac{1}{2}B^n \cdot B^{n*}, \quad \text{and} \quad A_i^n = (A_i \vee -n) \wedge n \quad, i = 1,2,...,d.$$

In view of $\|B^n\|^2 = \text{trace}\ \sigma^n$ and (2.8) we get

$$\|B^n\|^2 \leq d\,n, \quad n \in \mathbb{N}.$$

Also we obtain

$$| A^n | \leq \sqrt{dn}, \qquad \text{and} \qquad |A^n| \leq |A| \qquad \text{for all} \qquad n \in \mathbb{N}.$$

Furthermore, for every $z \in \mathbb{R}^d$

$$\langle\langle B^n z, z \rangle\rangle = \langle\langle U \Lambda_n^{\frac{1}{2}} U^* z, z \rangle\rangle = \langle\langle \Lambda_n^{\frac{1}{2}} U^* z, U^* z \rangle\rangle \geq n^{-\frac{1}{2}} |U^* z|^2 = n^{-\frac{1}{2}} |z|^2.$$

Therefore, the coefficients B^n and A^n satisfy the assumptions of Krylov's theorem (cf. [10], Theorem 2.6.1). Hence there exist continuous processes $(X^n, \mathbb{F}^n)$ and $(W^n, \mathbb{F}^n)$ defined on probability spaces $(\Omega^n, \mathcal{F}^n, \mathbf{P}^n)$ with filtrations $\mathbb{F}^n = (\mathcal{F}_t^n)_{t \geq 0}$ such that $(W^n, \mathbb{F}^n)$ are Wiener processes and, for any $n = 1, 2, \ldots$, the process (X^n, W^n) satisfies the equation

$$X_t^n = x_0 + \int_0^t B^n(s, X_s^n)\, dW_s^n + \int_0^t A^n(s, X_s^n) ds, \quad t \geq 0. \tag{3.1}$$

Set

$${}^m X_t^n := X_{t \wedge \tau_m(X^n)}^n, \quad {}^m Y_t^n := {}^m X_t^n - \int_0^t A^n(s, {}^m X_s^n) ds = x_0 + \int_0^t B^n(s, {}^m X_s^n)\, dW_s^n,$$

where τ_m is defined in (2.10) and we have used the property of stochastic integrals that $\int_0^{t\wedge\tau} H_s dW_s = \int_0^t H_{s\wedge\tau} dW_s$ for any stopping time τ with respect to a filtration $\mathbb{F}$, any Wiener process $(W, \mathbb{F})$ and a process $(H, \mathbb{F})$ such that the stochastic integral exists.

Obviously, the process ${}^m Y^n$ is a continuous martingale being the martingale part of the continuous semimartingale ${}^m X^n$. It follows (cf. Theorem I.4.52 in [9]) that

$$[{}^m X^{ni}, {}^m X^{nj}]_t = \langle {}^m Y^{ni}, {}^m Y^{nj} \rangle_t = \int_0^t \sigma_{ij}^n(s, {}^m X_s^n) ds \tag{3.2}$$

for all $m \in \mathbb{N}, i, j = 1, 2, \ldots, d$. Here $[\cdot, \cdot]$ and $\langle \cdot, \cdot \rangle$ denote the quadratic variation processes of a semimartingale and a martingale, respectively.

Now we are going to show that, for any fixed $m \in \mathbb{N}$, the sequence of bounded processes ${}^m X^n, n \geq 1$, is tight in $C([0, \infty), \mathbb{R}^d)$. Due to Aldous's criterion [1], for the tightness of the sequence $({}^m X^n)$ in the Skorokhod space $D([0, +\infty), \mathbb{R}^d)$, it suffices to show that for every sequence (τ^n) of $\mathbb{F}^n$-stopping times, every sequence (δ_n) of real numbers such that $\delta_n \downarrow 0$ and all $\varepsilon > 0$ it follows that

$$\lim_{n \to \infty} \mathbf{P}^n \Big(|{}^m X_{t \wedge (\tau^n + \delta_n)}^n - {}^m X_{t \wedge \tau^n}^n| > \varepsilon \Big) = 0. \tag{3.3}$$

Because the processes ${}^m X^n$ are continuous, from [9], Theorem VI.3.26, it follows that the tightness also holds in $C([0, +\infty), \mathbb{R}^d)$. We now verify (3.3). Let first n be fixed. By Chebyshev's inequality, Lemma 2.4, and Corollary 2.5 for any $L \geq 1$ we obtain

$$\mathbf{P}^n \Big(|{}^m X_{t \wedge (\tau^n + \delta_n)}^n - {}^m X_{t \wedge \tau^n}^n| > \varepsilon \Big)$$

$$\leq \mathbf{P}^n\Big(\Big| \int\limits_{t\wedge\tau^n}^{t\wedge(\tau^n+\delta_n)} B^n(s, {}^mX^n_s)dW^n_s \Big| > \frac{\varepsilon}{2}\Big) + \mathbf{P}^n\Big(\Big| \int\limits_{t\wedge\tau^n}^{t\wedge(\tau^n+\delta_n)} A^n(s, {}^mX^n_s)ds \Big| > \frac{\varepsilon}{2}\Big)$$

$$\leq 8\varepsilon^{-2}\,\mathbf{E}^n\left[\int\limits_{t\wedge\tau^n}^{t\wedge(\tau^n+\delta_n)} \text{trace}\ \sigma^n(s, {}^mX^n_s)\,ds\right] + 2\varepsilon^{-1}\mathbf{E}^n\left[\int\limits_{t\wedge\tau^n}^{t\wedge(\tau^n+\delta_n)} |\ A^n(s, {}^mX^n_s)\ |\ ds\right]$$

$$\leq 8\varepsilon^{-2}\,\mathbf{E}^n\left[\int\limits_{t\wedge\tau^n}^{t\wedge(\tau^n+\delta_n)} \text{trace}\ \sigma^n(s, {}^mX^n_s)\,ds\right] + 2\varepsilon^{-1}\mathbf{E}^n\left[\int\limits_{t\wedge\tau^n}^{t\wedge(\tau^n+\delta_n)} |\ A(s, {}^mX^n_s)\ |\ ds\right]$$

$$\leq L\delta_n(8\varepsilon^{-2} + 2\varepsilon^{-1}) + 8\varepsilon^{-2}\,\mathbf{E}^n\left[\int\limits_0^t 1_{\{\text{trace}\ \sigma^n > L\}}\text{trace}\ \sigma^n(s, {}^mX^n_s)\,ds\right] +$$

$$2\varepsilon^{-1}\mathbf{E}^n\left[\int\limits_0^t 1_{\{|A|>L\}}\ |\ A(s, {}^mX^n_s)\ |\ ds\right] \leq L\delta_n(8\varepsilon^{-2} + 2\varepsilon^{-1}) +$$

$$8\varepsilon^{-2}C\Big(\int\limits_{[0,t]\times U_m} 1_{\{\text{trace}\ \sigma^n > L\}}(\text{trace}\ \sigma^n)^{d+1}(\det\sigma^n)^{-1}dyds\Big)^{\frac{1}{d+1}}$$

$$+2\varepsilon^{-1}C\Big(\int\limits_{[0,t]\times U_m} 1_{\{|A|>L\}}\ |\ A\ |^{d+1}\ (\det\sigma^n)^{-1}dyds\Big)^{\frac{1}{d+1}}.$$

From (2.8) we get

$$d^{-1}\,\text{trace}\ \ \sigma^n \leq \lambda^n := \max_{i=1,\cdots,d}\lambda^n_i.$$

Clearly, $\det\sigma^n = \det\Lambda_n$ and using Lemma 2.3 and the obvious inequalities $\lambda^n_i \leq \lambda_i + 1$, $i = 1,\cdots,d$, we observe

$$(\text{trace}\ \ \sigma^n)^{d+1}\,(\det\sigma^n)^{-1} \leq d^{d+1}\,(\lambda+1)^{d+1}\,(\det\Lambda)^{-1}$$

where $\lambda := \max_{i=1,\cdots,d}\lambda_i$. Now $\det\Lambda = \det\sigma$ and, applying again inequalities (2.8), we obtain

$$(\text{trace}\ \ \sigma^n)^{d+1}\,(\det\sigma^n)^{-1} \leq d^{2(d+1)}\,(\max_{i=1,\cdots,d}(\sigma_{ii}) + 1)^{d+1}\,(\det\sigma)^{-1}. \tag{3.4}$$

Set $\gamma = \max_{i=1,\cdots,d}\sigma_{ii} + 1$. Lemma 2.3 implies then

$$\Big(\det\sigma^n\Big)^{-1} = \Big(\det\Lambda_n\Big)^{-1} = \Big(\prod_{i=1}^d \lambda^n_i\Big)^{-1} \leq 2^d \max_{i=1,\dots d}(\lambda_i+1)^d\Big(\prod_{i=1}^d \lambda_i\Big)^{-1} \leq$$

$$(2d)^d(\max_{i=1,\dots d}\sigma_{ii} + 1)^d(\det\sigma)^{-1} = (2d\gamma)^d(\det\sigma)^{-1}. \tag{3.5}$$

The estimates (3.4) and (3.5) yield together

$$\mathbf{P}^n\Big(|{}^mX^n_{t\wedge(\tau^n+\delta_n)} - {}^mX^n_{t\wedge\tau^n}| > \varepsilon\Big)$$

$$\le L\delta_n(8\varepsilon^{-2}+2\varepsilon^{-1}) + 8d^2\varepsilon^{-2}C\Big(\int\limits_{[0,t]\times U_m} 1_{\{\gamma>Ld^{-2}\}}\gamma^{d+1}(\det\sigma)^{-1}\,dy\,ds\Big)^{\frac{1}{d+1}}$$

$$+4Cd\varepsilon^{-1}\Big(\int\limits_{[0,t]\times U_m} 1_{\{|A|>L\}} \mid A\mid^{d+1}\gamma^d(\det\sigma)^{-1}dyds\Big)^{\frac{1}{d+1}}.$$

Thanks to $\mathbf{a_2}$) and $\mathbf{b}$) we know that the functions γ^{d+1} and $|A|^{d+1}\gamma^d$ are locally integrable with respect to $d\mu = (\det\sigma)^{-1}\,dy\,ds$ and, consequently, the right hand side converges to zero for $n\to\infty$ and then $L\to\infty$. This shows that $({}^mX^n)_{n\in\mathbb{N}}$ is tight in $C([0,+\infty),\mathbb{R}^d)$.

Using the theorem of Y. Prochorov (cf. [2], Theorem 6.1) and the diagonal method, we can choose a subsequence (n_k) and, for every $m\in\mathbb{N}$, probability measures $\tilde{\mathbf{R}}^m$ on $C([0,+\infty),\mathbb{R}^d)$ such that

$$\lim_{k\to\infty}\mathcal{D}_{\mathbf{P}^{n_k}}({}^mX^{n_k}) = \tilde{\mathbf{R}}^m.$$

For simplicity we assume that

$$\lim_{n\to\infty}\mathcal{D}_{\mathbf{P}^n}({}^mX^n) = \tilde{\mathbf{R}}^m \quad \text{for all} \quad m\in\mathbb{N}. \tag{3.6}$$

Let us extend $\tilde{\mathbf{R}}^m$ to probability measures $\mathbf{R}^m$ on $(E([0,+\infty),\bar{\mathbb{R}}^d),\mathcal{E}([0,+\infty),\bar{\mathbb{R}}^d))$ by

$$\mathbf{R}^m(A) = \tilde{\mathbf{R}}^m(A\cap C([0,+\infty),\mathbb{R}^d)), \quad A\in\mathcal{E}([0,+\infty),\bar{\mathbb{R}}^d). \tag{3.7}$$

We recall the definition of the coordinate mappings $Z=(Z_t)_{t\ge0}$ on $E([0,+\infty),\bar{\mathbb{R}}^d)$ by (2.2) and denote their restrictions to $C([0,+\infty),\mathbb{R}^d)$ by $\tilde{Z}=(\tilde{Z}_t)_{t\ge0}$. Similarly, let $\tilde{\tau}_a$ be the restrictions of the $\mathbb{E}$-stopping times τ_a defined in (2.10).

Let us prove that there exists a sequence of numbers $(a_m)_{m\ge1}$ such that $a_m\to\infty$ as $m\to\infty$ and $\tilde{\tau}_{a_m}(\cdot)$ is $\tilde{\mathbf{R}}^m$-a.s. continuous on $C([0,\infty),\mathbb{R}^d)$ for all $m=1,2,\ldots$ Because $C([0,\infty),\mathbb{R}^d)\subset D([0,\infty),\mathbb{R}^d)$, it suffices to show the continuity of $\tilde{\tau}_{a_m}(\cdot)$ on $D([0,\infty),\mathbb{R}^d)$ $\tilde{\mathbf{R}}^m$-a.s.

For any $m\in\mathbb{N}$ and $a\in(m-1,m)$ consider the function $J_a: D\to\bar{\mathbb{R}}$ defined by

$$J_a(z) := \inf\{t\ge0 : |z(t)|\ge a \text{ or } \int\limits_0^t |A(s,z(s))|ds \ge a \text{ or } \int\limits_0^t \text{trace}\,\sigma(s,z(s))ds\ge a\}.$$

It is easy to see that the function $a\to J_a(z)$ is increasing for any z hence there exists a countable set $N_m\subset(m-1,m)$ such that for all $a\notin N_m$

$$\tilde{\mathbf{R}}^m\Big\{z : \lim_{\varepsilon\downarrow0}J_{a-\varepsilon}(z) = \lim_{\varepsilon\downarrow0}J_{a+\varepsilon}(z)\Big\} = 1$$

and

$$\tilde{\mathbf{R}}^{m+1}\Big\{z : \lim_{\varepsilon\downarrow 0} J_{a-\varepsilon}(z) = \lim_{\varepsilon\downarrow 0} J_{a+\varepsilon}(z)\Big\} = 1.$$

By the definition of the Skorokhod topology, if $z_n \to z$ in D and $\lim_{\varepsilon\downarrow 0} J_{a-\varepsilon}(z) = \lim_{\varepsilon\downarrow 0} J_{a+\varepsilon}(z)$, then $J_a(z_n) \to J_a(z)$ as $n \to \infty$ (cf. [8], Section 2.7). Clearly, the function $z \to J_a(z)$ is continuous for $\tilde{\mathbf{R}}^m$- and $\tilde{\mathbf{R}}^{m+1}$-almost every z for all $a \in (0,\infty) \setminus \cup_{m=1}^{\infty} N_m$. Hence there is a sequence $(a_m)_{m\in\mathbb{N}}$ with $a_m \in (m-1, m]$ such that $\tilde{\tau}_{a_m}(\cdot)$ are $\tilde{\mathbf{R}}^m$-a.s. and also $\tilde{\mathbf{R}}^{m+1}$-a.s. continuous functions on $C([0,+\infty), \mathbb{R}^d)$.

Let us introduce the continuous processes ${}^{a_m}X^n$ and ${}^{a_m}Z$ defined on $(\Omega^n, \mathcal{F}^n, \mathbf{P}^n)$ and $(E([0,+\infty), \bar{\mathbb{R}}^d), \mathcal{E}([0,+\infty), \bar{\mathbb{R}}^d))$, respectively, by

$${}^{a_m}X^n_t = X^n_{t\wedge\tau_{a_m}(X^n)} \quad \text{and} \quad {}^{a_m}Z_t = Z_{t\wedge\tau_{a_m}(Z)}, \quad t \geq 0.$$

To simplify the notation, from now on we write

$${}^{m}X^n_t = {}^{a_m}X^n_t, \quad {}^{m}Z_t = {}^{a_m}Z_t.$$

The restriction of ${}^mZ$ to $C([0,+\infty), \mathbb{R}^d)$ is denoted by ${}^m\tilde{Z}$. In the sequel, as image space for ${}^mX^n$ and ${}^mZ$ we consider $(C([0,+\infty), \mathbb{R}^d), \mathcal{C}([0,+\infty), \mathbb{R}^d))$. We define the probability measures $\tilde{\mathbf{Q}}^m$ by

$$\tilde{\mathbf{Q}}^m = \mathcal{D}_{\mathbf{R}^m}({}^mZ) \;\; (= \mathcal{D}_{\tilde{\mathbf{R}}^m}({}^m\tilde{Z})),$$

i.e., $\tilde{\mathbf{Q}}^m$ on $(C([0,+\infty), \mathbb{R}^d), \mathcal{C}([0,+\infty), \mathbb{R}^d))$ is the distribution of the stopped process ${}^mZ$ defined on the probability space $(E([0,+\infty), \bar{\mathbb{R}}^d), \mathcal{E}([0,+\infty), \bar{\mathbb{R}}^d), \mathbf{R}^m)$. The probability measures $\mathbf{Q}^m$ on $(E([0,+\infty), \bar{\mathbb{R}}^d), \mathcal{E}([0,+\infty), \bar{\mathbb{R}}^d))$ are now introduced as the extensions of $\tilde{\mathbf{Q}}^m$ analogously to (3.7).

Given that $(a_m)_{m\in\mathbb{N}}$ is chosen such that $\tau_{a_m}(\cdot)$ is continuous $\mathbf{R}^m$-a.s. and $\mathbf{R}^{m+1}$-a.s., the relation (3.6) and the continuous mapping theorem (cf. [2], Theorem 5.1) imply

$$\lim_{n\to\infty} \mathcal{D}_{\mathbf{P}^n}({}^mX^n) = \tilde{\mathbf{Q}}^m \tag{3.8}$$

and

$$\lim_{n\to\infty} \mathcal{D}_{\mathbf{P}^n}({}^mX^n) = \mathcal{D}_{\mathbf{R}^{m+1}}({}^mZ) \quad \text{for all} \quad m \in \mathbb{N}.$$

This yields the equality

$$\mathcal{D}_{\mathbf{R}^{m+1}}({}^mZ) = \tilde{\mathbf{Q}}^m \quad \text{for all} \quad m \in \mathbb{N}. \tag{3.9}$$

We now state the following

Lemma 3.2 *For all $m \in \mathbb{N}$ we have*

1) $\mathcal{D}_{\mathbf{Q}^m}({}^mZ) = \tilde{\mathbf{Q}}^m$.

2) $\mathcal{D}_{\mathbf{Q}^{m+1}}({}^mZ) = \mathcal{D}_{\mathbf{Q}^m}({}^mZ)$.

3) $\mathbf{Q}^{m+1}(A) = \mathbf{Q}^m(A)$ *for all* $A \in \mathcal{E}_{\tau_{a_m}}$.

Proof. 1) follows from the identity ${}^mZ \circ {}^mZ = {}^mZ$ and the definition of $\tilde{\mathbf{Q}}^m$ and $\mathbf{Q}^m$. Using ${}^mZ \circ {}^{m+1}Z = {}^mZ$ and relation (3.9) we observe

$$\mathcal{D}_{\mathbf{Q}^{m+1}}({}^mZ) = \mathcal{D}_{\mathbf{R}^{m+1}}({}^mZ) = \tilde{\mathbf{Q}}^m$$

which implies 2) in view of 1). Finally, 3) is a simple consequece of 2) and the property ${}^mZ^{-1}(A) = A$ for every $A \in \mathcal{C}_{\tilde{\tau}_{a_m}} := \mathcal{E}_{\tau_{a_m}} \cap \mathcal{C}([0,+\infty), \mathbb{R}^d)$. This proves Lemma 3.2. □

In view of Lemma 3.2 and Proposition 2.1 there exists a unique probability measure $\mathbf{Q}$ on $(E([0,+\infty), \bar{\mathbb{R}}^d), \mathcal{E}([0,+\infty), \bar{\mathbb{R}}^d))$ such that

$$\mathbf{Q}(A) = \mathbf{Q}^m(A) \quad \text{for all} \quad A \in \mathcal{E}_{\tau_{a_m}}, \quad m \in \mathbb{N}.$$

The definition of $\mathbf{Q}$, the representation (2.4) and Lemma 3.2 yield

$$\mathcal{D}_{\mathbf{Q}}({}^mZ) = \mathcal{D}_{\mathbf{Q}^m}({}^mZ) = \tilde{\mathbf{Q}}^m.$$

Hence statement (3.8) can be rewritten as

$$\lim_{n\to\infty} \mathcal{D}_{\mathbf{P}^n}({}^mX^n) = \mathcal{D}_{\mathbf{Q}}({}^mZ) \quad \text{for all} \quad m \in \mathbb{N}. \tag{3.10}$$

Thus we constructed processes ${}^mZ$ on $(E([0,+\infty), \bar{\mathbb{R}}^d), \mathcal{E}([0,+\infty), \bar{\mathbb{R}}^d), \mathbf{Q})$ to which the sequence $({}^mX^n)_{n\in\mathbb{N}}$ converges weakly. We also notice that the process ${}^mZ$ is a continuous semimartingale with respect to $\mathbb{F}^Z$ (cf. Theorem VI.6.1 and Remark VI.6.5 in [9]). Hence the process Z is a continuous semimartingale with respect to $\mathbb{F}^Z$ up to the explosion time $\tau_\Delta(Z)$. Our aim is to show that the process Z defined on the probability space $(E([0,+\infty), \bar{\mathbb{R}}^d), \mathcal{E}^{\mathbf{Q}}([0,+\infty), \bar{\mathbb{R}}^d), \mathbf{Q})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, $\mathcal{F}_t = \mathcal{E}^{\mathbf{Q}}_{t+}$, $t \geq 0$, where the superscript $\mathbf{Q}$ means completion in $\mathcal{E}^{\mathbf{Q}}([0,+\infty), \bar{\mathbb{R}}^d)$, is the desired solution of the equation (1.1).

In order to prove this, it suffices to show that there exists a Wiener process W such that

$$Z_t = x_0 + \int_0^t B(s, Z_s)dW_s + \int_0^t A(s, Z_s)ds \quad \text{on} \quad \{t < \tau_\Delta\} \quad \mathbf{Q}\text{-a.s.}$$

or, equivalently,

$${}^mZ_t = x_0 + \int_0^t B(s, {}^mZ_s)dW_s + \int_0^t A(s, {}^mZ_s)ds \qquad \mathbf{Q}\text{-a.s.}$$

for all $t \geq 0$ and $m \in \mathbb{N}$.

According to the well-known theorem of J. Doob (cf. Theorem 7.1′ in [6]), we need then only to verify that, for all $i, j = 1, 2, \ldots, d, m \in \mathbb{N}$, and $t \geq 0$, it holds

$$\langle {}^mY^i, {}^mY^j \rangle_t = \int_0^t \sigma_{ij}(s, {}^mZ_s)ds, \quad \mathbf{Q}\text{-a.s.} \tag{3.11}$$

where ${}^{m}Y$ is a continuous martingale of the form

$$ {}^{m}Y_t = {}^{m}Z_t - \int_0^t A(s, {}^{m}Z_s) ds. $$

If we show that

$$ \lim_{n\to\infty} \mathcal{D}_{\mathbf{P}^n}\Big(\langle {}^{m}Y^{ni}, {}^{m}Y^{nj}\rangle, \int_0^{\cdot} \sigma_{ij}^n(s, {}^{m}X_s^n)\, ds\Big) = \mathcal{D}_{\mathbf{Q}}\Big(\langle {}^{m}Y^{i}, {}^{m}Y^{j}\rangle, \int_0^{\cdot} \sigma_{ij}(s, {}^{m}Z_s)\, ds\Big) \quad (3.12) $$

for all $m \in \mathbb{N}, i, j = 1, 2, \ldots, d$, we will be able to finish the proof.
Indeed, because of the relation (3.2), an application of the continuous mapping theorem (cf. [2], Theorem 5.1) to the functional ρ implies then

$$ \rho\Big(\langle {}^{m}Y^{i}, {}^{m}Y^{j}\rangle, \int_0^{\cdot} \sigma_{ij}(s, {}^{m}Z_s) ds\Big) = 0 $$

which verifies (3.11).

Let us prove (3.12). For this we will need the following lemma stating that the sequence of processes $({}^{m}Y^n), n \geq 1$, converges weakly to the process ${}^{m}Y$ for any $m \in \mathbb{N}$.

Lemma 3.3 *For any $m \in \mathbb{N}$, ${}^{m}Z - \int_0^{\cdot} A(s, {}^{m}Z_s) ds$ is a continuous martingale and*

$$ \lim_{n\to\infty} \mathcal{D}_{\mathbf{P}^n}\Big({}^{m}X^n - \int_0^{\cdot} A^n(s, {}^{m}X_s^n) ds\Big) = \mathcal{D}_{\mathbf{Q}}\Big({}^{m}Z - \int_0^{\cdot} A(s, {}^{m}Z_s) ds\Big). \quad (3.13) $$

Proof. First we fix $p \in \mathbb{N}$ and show

$$ \lim_{n\to\infty} \mathcal{D}_{\mathbf{P}^n}\Big({}^{m}X^n - \int_0^{\cdot} A^p(s, {}^{m}X_s^n)\, ds\Big) = \mathcal{D}_{\mathbf{Q}}\Big({}^{m}Z - \int_0^{\cdot} A^p(s, {}^{m}Z_s)\, ds\Big) \quad (3.14) $$

for all $m \in \mathbb{N}$. Since $|A^p|$ is bounded by p and hence $(A^p)^{d+1}$ is locally integrable we can choose a sequence $(f_k)_{k\in\mathbb{N}}$ of continuous functions uniformly bounded by p such that

$$ \lim_{k\to\infty} \int_{[0,N]\times U_m} |f_k - A^p|^{d+1}\, dy\, ds = 0 $$

for all $m, N \in \mathbb{N}$.

Consider the functional F_k on $C([0,\infty), \mathbb{R}^d)$ defined as

$$ F_k(x(t)) := x(t) - \int_0^t f_k(s, x(s)) ds. $$

Clearly, for any $k \in \mathbb{N}$, the functional F_k is continuous. Hence applying the continuous mapping theorem (cf. [2], Theorem 5.1) to the functional F_k, we obtain

$$\lim_{n\to\infty} \mathcal{D}_{\mathbf{P}^n}\Big({}^mX^n - \int_0^{\cdot} f_k(s, {}^mX^n_s)\,ds\Big) = \mathcal{D}_{\mathbf{Q}}\Big({}^mZ - \int_0^{\cdot} f_k(s, {}^mZ_s)\,ds\Big). \tag{3.15}$$

Next we extend Krylov's estimates to the limit process ${}^mZ$.

Lemma 3.4 *For any $m \in \mathbb{N}, t > 0$ and any nonnegative measurable function f, we have*

$$\mathbf{E}_{\mathbf{Q}}\left[\int_0^t f(s, {}^mZ_s)\,ds\right] \le C\left(\int_{[0,t]\times U_m} f^{d+1}(s,y)\ [\det\sigma(s,y)]^{-1}\ dy\,ds\right)^{\frac{1}{d+1}}.$$

Proof. First we notice that it suffices to verify the above inequality only for any continuous and bounded function f. Indeed, using then the standard arguments of the monotone class theorem (cf. Lemma 2.4), the inequality would follow for any nonnegative measurable function f.

Because of the assumption $\mathbf{a_1}$) of Theorem 3.1, it follows that $\det\sigma(s,y) \neq 0$ for almost all $(s,y) \in [0,t] \times U_m$. Hence using Corollary 2.5, the continuous mapping theorem and Fatou's Lemma, we obtain

$$\begin{aligned}
\mathbf{E}_{\mathbf{Q}}\left[\int_0^t f(s, {}^mZ_s)\,ds\right] &\le \liminf_{n\to\infty} \mathbf{E}^n\left[\int_0^t f(s, {}^mX^n{}_s)\,ds\right] \\
&\le C\liminf_{n\to\infty}\left(\int_{[0,t]\times U_m} f^{d+1}(s,y)\ [\det\sigma^n(s,y)]^{-1}\ dy\,ds\right)^{\frac{1}{d+1}} \\
&= C\left(\int_{[0,t]\times U_m} f^{d+1}(s,y)\ [\det\sigma(s,y)]^{-1}\ dy\,ds\right)^{\frac{1}{d+1}}
\end{aligned}$$

where we used the fact that $(\det\sigma^n)^{-1}$ is uniformly integrable over $[0,t]\times U_m$ because of the estimate (3.5) and the conditions $\mathbf{a_1}$) and $\mathbf{a_2}$) of Theorem 3.1. □

Using Lemma 3.4 and the Chebyshev's inequality we estimate

$$\begin{aligned}
&\mathbf{Q}\Big(\sup_{0\le t\le N} \Big|\int_0^t f_k(s, {}^mZ_s)\,ds - \int_0^t A^p(s, {}^mZ_s)\,ds\Big| > \varepsilon\Big) \\
&\le\ \ \varepsilon^{-1}\,\mathbf{E}_{\mathbf{Q}}\left[\int_0^N |f_k(s, {}^mZ_s) - A^p(s, {}^mZ_s)|\,ds\right] \\
&\le\ \ \varepsilon^{-1}C\Big(\int_{[0,N]\times U_m} |f_k - A^p|^{d+1}\,[\det\sigma]^{-1}\,dy\,ds\Big)^{\frac{1}{d+1}}.
\end{aligned}$$

Since $|f_k - A^p|^{d+1}$ is bounded by $(2p)^{d+1}$ uniformly for all k, the condition $\mathbf{a_1}$) yields that the right hand side converges to zero as $k \to \infty$. Hence

$$\lim_{k\to\infty} \mathcal{D}_{\mathbf{Q}}\Big({}^mZ - \int_0^{\cdot} f_k(s, {}^mZ_s)\,ds\Big) = \mathcal{D}_{\mathbf{Q}}\Big({}^mZ - \int_0^{\cdot} A^p(s, {}^mZ_s)\,ds\Big). \tag{3.16}$$

In the next step we estimate

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbf{P}^n\Big(\sup_{0\le t\le N} |\int_0^t f_k(s, {}^mX^n_s)\, ds - \int_0^t A^p(s, {}^mX^n_s)\, ds| > \varepsilon\Big) \\
&\le\ \varepsilon^{-1} \limsup_{n\to\infty} \mathbf{E}^n \left[\int_0^N |f_k(s, {}^mX^n_s) - A^p(s, {}^mX^n_s)|\, ds\right] \\
&\le\ \varepsilon^{-1} C \limsup_{n\to\infty}\Big(\int_{[0,N]\times U_m} |f_k - A^p|^{d+1} \,[\det \sigma^n]^{-1}\, dy\, ds\Big)^{\frac{1}{d+1}}.
\end{aligned}$$

Now, since $\lim_{k\to\infty} f_k = A^p$ a.e., f_k is bounded by p as well as A^p, and $(\det \sigma^n)^{-1}$ is uniformly integrable over $[0, N] \times U_m$ because of $\mathbf{a_2}$) and the estimate (3.5), the right hand side converges to zero as $k \to \infty$. Therefore, we have

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mathbf{P}^n\Big(|\int_0^t f_k(s, {}^mX^n_s)\, ds - \int_0^t A^p(s, {}^mX^n_s)\, ds| > \varepsilon\Big) = 0 \tag{3.17}$$

for all $\varepsilon > 0$.

Applying Proposition 2.2, from relations (3.15), (3.16) and (3.17) we obtain the weak convergence (3.14).

Next we observe that

$$\lim_{p\to\infty} \mathcal{D}_{\mathbf{Q}}\Big({}^mZ - \int_0^{\cdot} A^p(s, {}^mZ_s)\, ds\Big) = \mathcal{D}_{\mathbf{Q}}\Big({}^mZ - \int_0^{\cdot} A(s, {}^mZ_s)\, ds\Big) \tag{3.18}$$

because in view of Lemma 3.4

$$\begin{aligned}
&\mathbf{Q}\Big(\sup_{0\le t\le N} |\int_0^t A^p(s, {}^mZ_s)\, ds - \int_0^t A(s, {}^mZ_s) ds| > \varepsilon\Big) \\
&\le\ \varepsilon^{-1} \mathbf{E}_{\mathbf{Q}} \left[\int_0^N |A^p(s, {}^mZ_s) - A(s, {}^mZ_s)|\, ds\right] \\
&\le\ \varepsilon^{-1} C \Big(\int_{[0,N]\times U_m} |A^p - A|^{d+1} \,[\det \sigma]^{-1}\, dy\, ds\Big)^{\frac{1}{d+1}}.
\end{aligned}$$

By the Lebesgue theorem of uniform convergence, the right term converges to zero as $p \to \infty$ since by the definition of A^p it follows $\lim_{p\to\infty} |A^p - A|^{d+1} = 0$ a.e. and, in view of the condition $\mathbf{b}$), the sequence $(|A^p - A|^{d+1}\, [\det \sigma]^{-1})_{p\ge 1}$ is uniformly integrable over $[0, N] \times U_m$.

Finally, it can be easily seen that

$$\lim_{n\to\infty} |A^n - A^p|^{d+1} \,[\det \sigma^n]^{-1} = |A - A^p|^{d+1} \,[\det \sigma]^{-1} \text{ a.e.}$$

Moreover, the estimate (3.5) and the condition **b**) of the Theorem 3.1 imply that the sequence $(|A^n - A^p|^{d+1}\,[\det\sigma^n]^{-1})_{n\geq 1}$ is uniformly integrable over $[0,N]\times U_m$. Thus, by Corollary 2.5 and the Lebesgue theorem of uniform convergence we have

$$\begin{aligned}
\Delta_p &:= \limsup_{n\to\infty}\ \mathbf{P}^n\Big(\sup_{0\leq t\leq N}|\int_0^t A^n(s,{}^mX^n_s)ds - \int_0^t A^p(s,{}^mX^n_s)ds| > \varepsilon\Big) \\
&\leq\ \varepsilon^{-1}\ \limsup_{n\to\infty}\ \mathbf{E}_n\left[\int_0^N |A^n(s,{}^mX^n_s) - A^p(s,{}^mX^n_s)|\,ds\right] \\
&\leq\ \varepsilon^{-1}C\ \limsup_{n\to\infty}\ \left(\int_{[0,N]\times U_m} |A^n - A^p|^{d+1}\ [\det\sigma^n]^{-1}\ dy\,ds\right)^{\frac{1}{d+1}} \\
&\leq\ \varepsilon^{-1}C\ \left(\int_{[0,N]\times U_m} |A - A^p|^{d+1}\,[\det\sigma]^{-1}\ dyds\right)^{\frac{1}{d+1}}.
\end{aligned}$$

By the same arguments as above, $\lim_{p\to\infty}\Delta_p = 0$. Consequently

$$\lim_{p\to\infty}\limsup_{n\to\infty}\ \mathbf{P}^n\Big(\sup_{0\leq t\leq N}|\int_0^t A^n(s,{}^mX^n_s)ds - \int_0^t A^p(s,{}^mX^n_s)ds| > \varepsilon\Big) = 0 \tag{3.19}$$

for all $\varepsilon > 0$. Finally, using Proposition 2.2, from (3.14), (3.18) and (3.19) we obtain (3.13). The process ${}^mY$ is a continuous martingale with respect to the filtration $\mathbb{F}^{{}^mZ}$ as a weak limit of continuous martingales ${}^mY^n$ (cf. [9], Proposition IX.1.10). This finishes the proof of Lemma 3.3. □

As a consequence of Lemma 3.3 and Corollary VI.6.6 in [9], we obtain

Corollary 3.5 *For any $m\in\mathbb{N}$ and $i,j=1,2,\ldots,d$, it holds*

$$\lim_{n\to\infty}\mathcal{D}_{\mathbf{P}^n}\Big(\Big\langle{}^mY^{ni},{}^mY^{nj}\Big\rangle\Big) = \mathcal{D}_{\mathbf{Q}}\Big(\Big\langle{}^mZ^i - \int_0^{\cdot} A_i(s,{}^mZ_s)ds,{}^mZ^j - \int_0^{\cdot} A_j(s,{}^mZ_s)ds\Big\rangle\Big). \tag{3.20}$$

Lemma 3.6 *For any $m\in\mathbb{N}$ and $i,j=1,2,\ldots,d$, we have*

$$\lim_{n\to\infty}\mathcal{D}_{\mathbf{P}^n}\Big([{}^mX^{ni},{}^mX^{nj}],\int_0^{\cdot}\sigma^n_{ij}(s,{}^mX^n_s)ds\Big) = \mathcal{D}_{\mathbf{Q}}\Big([{}^mZ^i,{}^mZ^j],\int_0^{\cdot}\sigma_{ij}(s,{}^mZ_s)ds\Big). \tag{3.21}$$

Sketch of the proof : We follow the same steps and use similar arguments as in the proof of Lemma 3.3. Using Proposition 2.2, for proving (3.21), it suffices to verify that, for all $m\in\mathbb{N}$ and $i,j=1,2,\ldots,d$, it holds

$$\lim_{n\to\infty}\mathcal{D}_{\mathbf{P}^n}\Big([{}^mX^{ni},{}^mX^{nj}],\int_0^{\cdot}\sigma^p_{ij}(s,{}^mX^n_s)ds\Big) = \mathcal{D}_{\mathbf{Q}}\Big([{}^mZ^i,{}^mZ^j],\int_0^{\cdot}\sigma^p_{ij}(s,{}^mZ_s)ds\Big) \tag{3.22}$$

for any fixed $p \in \mathbb{N}$;

$$\lim_{p\to\infty} \mathcal{D}_{\mathbf{Q}}\Big([{}^mZ^i,{}^mZ^j], \int_0^{\cdot} \sigma_{ij}^p(s,{}^mZ_s)ds\Big) = \mathcal{D}_{\mathbf{Q}}\Big([{}^mZ^i,{}^mZ^j], \int_0^{\cdot} \sigma_{ij}(s,{}^mZ_s)ds\Big); \qquad (3.23)$$

$$\lim_{p\to\infty}\lim_{n\to\infty} \mathbf{P}^n\Big(\sup_{0\le t\le N} |\int_0^t \sigma_{ij}^n(s,{}^mX_s^n)ds - \int_0^t \sigma_{ij}^p(s,{}^mX_s^n)ds| > \varepsilon\Big) = 0 \qquad (3.24)$$

for all $\varepsilon > 0$ and $N \in \mathbb{N}$.

To prove (3.22) we notice that, for any fixed $p \in \mathbb{N}$ and $i, j = 1, 2, \dots, d$, there exists a sequence of continuous bounded functions $(f_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \int_{[0,N]\times U_m} |f_k - \sigma_{ij}^p|^{d+1} ds dy = 0$$

for all $m, N \in \mathbb{N}$. The later follows from the definition of σ_{ij}^p and the conditions of Theorem 3.1. Now, for any $k \in \mathbb{N}$, the functional G_k defined as

$$G_k(x(t), y(t)) := \Big(x(t), \int_0^t f_k(s, y(s))ds\Big).$$

is continuous on $C([0,\infty),\mathbb{R}^d) \times C([0,\infty),\mathbb{R}^d)$.

Further we remark that from the weak convergence of the sequence of continuous bounded semimartingales ${}^mX^n$ to the continuous semimartingale ${}^mZ$ as $n \to \infty$ it follows (cf. [9], Theorem VI.6.1) that, for any $i, j = 1, 2, \dots, d$, the sequence of vectors $([{}^mX^{ni}, {}^mX^{nj}], {}^mX^n)$ converges weakly to the vector $([{}^mZ^i, {}^mZ^j], {}^mZ)$. Hence applying the continuous mapping theorem to the functional G_k we obtain

$$\lim_{n\to\infty} \mathcal{D}_{\mathbf{P}^n}\Big([{}^mX^{ni},{}^mX^{nj}], \int_0^{\cdot} f_k(s,{}^mX_s^n)ds\Big) = \mathcal{D}_{\mathbf{Q}}\Big([{}^mZ^i,{}^mZ^j], \int_0^{\cdot} f_k(s,{}^mZ_s)ds\Big) \qquad (3.25)$$

for any $k \in \mathbb{N}$.

Using Krylov's estimates for the limit process ${}^mZ$ (cf. Lemma 3.4) we obtain that, for any $\varepsilon > 0, N \in \mathbb{N}$,

$$\lim_{k\to\infty} \mathbf{Q}\Big(\rho^2\Big(([{}^mZ^i,{}^mZ^j], \int_0^{\cdot} f_k(s,{}^mZ_s)ds); ([{}^mZ^i,{}^mZ^j], \int_0^{\cdot} \sigma_{ij}^p(s,{}^mZ_s)ds)\Big) > \varepsilon\Big) = 0$$

which leads to

$$\lim_{k\to\infty} \mathcal{D}_{\mathbf{Q}}\Big([{}^mZ^i,{}^mZ^j], \int_0^{\cdot} f_k(s,{}^mZ_s)ds\Big) = \mathcal{D}_{\mathbf{Q}}\Big([{}^mZ^i,{}^mZ^j], \int_0^{\cdot} \sigma_{ij}^p(s,{}^mZ_s)ds\Big) \qquad (3.26)$$

for all $m \in \mathbb{N}, i, j = 1, 2, \ldots, d$.

Similarly, using Krylov's estimates, the relation (3.5), and the conditions of Theorem 3.1, we can verify that

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mathbf{P}^n\Big(\sup_{0\le t\le N} |\int_0^t f_k(s, {}^mX^n{}_s)ds - \int_0^t \sigma^p_{ij}(s, {}^mX^n{}_s)ds| > \varepsilon\Big) = 0 \tag{3.27}$$

for all $\varepsilon > 0, N \in \mathbb{N}, m \in \mathbb{N}$, and $i, j = 1, 2, \ldots, d$.

The relations (3.25), (3.26), (3.27), and the Proposition 2.2 imply (3.22).

For the relation (3.23) to be true, it is enough to verify that, for any $\varepsilon > 0, m \in \mathbb{N}, N \in \mathbb{N}$, and $i, j = 1, 2, \ldots, d$, it holds

$$\lim_{p\to\infty} \mathbf{Q}\Big(\sup_{0\le t\le N} |\int_0^t \sigma^p_{ij}(s, {}^mZ_s)ds - \int_0^t \sigma_{ij}(s, {}^mZ_s)ds| > \varepsilon\Big) = 0. \tag{3.28}$$

By the Chebyshev's inequality and Lemma 3.4

$$\mathbf{Q}\Big(\sup_{0\le t\le N} |\int_0^t \sigma^p_{ij}(s, {}^mZ_s)ds - \int_0^t \sigma_{ij}(s, {}^mZ_s)ds| > \varepsilon\Big)$$

$$\le \varepsilon^{-1}C\Big(\int_{[0,N]\times U_m} |\sigma^p_{ij} - \sigma_{ij}|^{d+1}[\det\sigma]^{-1}dyds\Big)^{\frac{1}{d+1}}.$$

Due to the estimate (3.5) and the condition $\mathbf{a_2}$) of Theorem 3.1, the sequence of functions $(|\sigma^p_{ij} - \sigma_{ij}|^{d+1}[\det\sigma]^{-1})_{p\ge1}$ is uniformly integrable over any set $[0, N] \times U_m$ and the relation (3.28) is true because of the Lebesgue theorem of uniform convergence.

Finally, to show (3.24) we can use the Chebyshev's inequality, Lemma 2.3, and Lemma 2.4 to estimate

$$\begin{aligned}
&\lim_{p\to\infty}\lim_{n\to\infty} \mathbf{P}^n\Big(\sup_{0\le t\le N} |\int_0^t \sigma^n_{ij}(s, {}^mX^n_s)\,ds - \int_0^t \sigma^p_{ij}(s, {}^mX^n_s)\,ds| > \varepsilon\Big)\\
\le\ & \varepsilon^{-1} \lim_{p\to\infty} \limsup_{n\to\infty} \mathbf{E}_n \int_0^N |\sigma^n_{ij}(s, {}^mX^n_s) - \sigma^p_{ij}(s, {}^mX^n_s)|\,ds\\
\le\ & \varepsilon^{-1}C \lim_{p\to\infty} \limsup_{n\to\infty} \Big(\int_{[0,N]\times U_m} |\sigma^n_{ij} - \sigma^p_{ij}|^{d+1}\,[\det\sigma^n]^{-1}\,dy\,ds\Big)^{\frac{1}{d+1}}\\
\le\ & \varepsilon^{-1}C \lim_{p\to\infty} \limsup_{n\to\infty} \Big(\int_{[0,N]\times U_m} \max_{k=1,2,\cdots,d} |\lambda^n_k - \lambda^p_k|^{d+1}\,[\det\sigma^n]^{-1}\,dy\,ds\Big)^{\frac{1}{d+1}}
\end{aligned}$$

$$\leq \varepsilon^{-1} C \lim_{p\to\infty} \limsup_{n\to\infty} \Bigg(\int\limits_{[0,N]\times U_m} \max_{k=1,2,\cdots,d} (\lambda_k^n)^{d+1} 1_{\{\lambda_k > p\}} [\det \sigma^n]^{-1} \, dyds$$

$$+ \int\limits_{[0,N]\times U_m} \max_{k=1,2,\cdots,d} p^{-(d+1)} 1_{\{\lambda_k \leq \frac{1}{p}\}} [\det \sigma^n]^{-1} \, dyds \Bigg)^{\frac{1}{d+1}}$$

$$\leq \varepsilon^{-1} C \lim_{p\to\infty} \Bigg(\int\limits_{[0,N]\times U_m} \max_{k=1,2,\cdots,d} (\lambda_k + 1)^{d+1} 1_{\{\lambda_k > p\}} [\det \sigma]^{-1} \, dyds$$

$$+ p^{-(d+1)} 2^d \int\limits_{[0,N]\times U_m} \max_{k=1,2,\cdots,d} (\lambda_k + 1)^{d+1} [\det \sigma]^{-1} \, dyds \Bigg)^{\frac{1}{d+1}}.$$

Because of the condition $\mathbf{a_2}$), the function

$$\max_{k=1,2,\cdots,d} (\lambda_k + 1)^{d+1} [\det \sigma]^{-1}$$

is integrable over $[0, N] \times U_m$ hence the right hand side of the last inequality is equal to zero. □

Now we return to the proof of Theorem 3.1. Notice that the process $[{}^m Z^i, {}^m Z^j]$ is predictable as a continuous process and it is uniquely determined as the weak limit of the sequence of processes $[{}^m X^{ni}, {}^m X^{nj}] = \langle {}^m Y^{ni}, {}^m Y^{nj} \rangle$ (cf. [9], Theorem IX.2.4). Hence using Corollary 3.5 we conclude that $[{}^m Z^i, {}^m Z^j] = \langle {}^m Y^i, {}^m Y^j \rangle$ and, by Lemma 3.6, the relation (3.12) is verified. The proof of Theorem 3.1 is finished.□

Corollary 3.7 *Suppose the coefficients A and B satisfy the following conditions:*

- $\mathbf{a_1}$) $(\det B \cdot B^*)^{-1} \in L^{\rm loc}([0, +\infty) \times \mathbb{R}^d)$.
- **c**) *There exists $p > 1$ such that $|A|^{p(d+1)} \in L^{\rm loc}([0, +\infty) \times \mathbb{R}^d, \mu)$ and $\|B\|^{2q(d+1)} \in L^{\rm loc}([0, +\infty) \times \mathbb{R}^d, \mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

Then, for an arbitrary $x_0 \in \mathbb{R}^d$, there exists a solution X of the equation (1.1) with $X_0 = x_0$.

The condition imposed on the coefficient A in Corollary 3.7 can be formulated as "There exists a number $p > d + 1$ such that $|A|^p \in L^{\rm loc}([0, +\infty) \times \mathbb{R}^d, \mu)$".

It is worthwhile to compare this condition with the integrability condition on A found by N.I. Portenko [16]. He constructed a (non-exploding) solution of (1.1) assuming the matrix B to be uniformly continuous, bounded and uniformly nondegenerate, and the drift coefficient A to be integrable on $[0, T] \times \mathbb{R}^d$ of order $p > d + 2$ with respect to Lebesgue measure for any $T > 0$. In other words, he assumed the global Lebesgue integrability in space variable. In our case we assume the local integrability of order $p > d + 1$ but with respect to the measure μ. In particular, assume the matrix B to be nondegenerate as is the case in Portenko's conditions. Then there exists a constant $c > 0$ such that

$\det BB^* > c$ and $[\det BB^*]^{-1} < \frac{1}{c} < \infty$. Therefore, our condition on drift becomes "there exists a number $p > d+1$ such that $|A|^p$ is locally integrable with respect to the Lebesgue measure".

Remark 3.8 *Assume $d = 1$ and $A = 0$. Then our conditions coincide with the existence conditions found by T. Senf [21] and A. Rozkosz and L. Slomonski [18] for (possibly, exploding) solutions of (1.1).*

Remark 3.9 *Consider the case of a one-dimensional ($d = 1$) equation (1.1) with the unit diffusion coefficient $B = 1$. Then the existence condition found becomes $A^2 \in L^{loc}([0,\infty) \times \mathbb{R})$ which coincides with the known existence condition for such SDEs stated first in [13].*

Theorem 3.1 can be extended in the following way. We introduce the sets

$$\mathcal{N} = \{(t,x) \in [0,+\infty) \times \mathbb{R}^d : \; B(s,x) = 0 \quad \text{and} \quad A(s,x) = 0 \quad \text{for almost all} \quad s \geq t\}$$

and

$$\mathcal{M} = \{(t,x) \in [0,+\infty) \times \mathbb{R}^d : \; \mu(S_\delta(t,x)) = \infty \quad \text{for all} \quad \delta > 0\}$$

where $S_\delta(t,x)$ denotes the ball with center (t,x) and radius δ in $[0,+\infty) \times \mathbb{R}^d$ and the measure μ is defined in (1.4). Clearly, $\mathcal{M}^c := [0,+\infty) \times \mathbb{R}^d \setminus \mathcal{M}$ is an open subset of $[0,+\infty) \times \mathbb{R}^d$ and μ is a locally finite measure on $\mathcal{M}^c$, i.e., $\mu(K) < +\infty$ for every compact subset of $\mathcal{M}^c$.

Theorem 3.10 *Suppose the following conditions are satisfied:*

- $\mathbf{a_1^*}$) $\mathcal{M} \subseteq \mathcal{N}$.
- $\mathbf{a_2^*}$) $\|B\|^{2(d+1)} \in L^{\text{loc}}(\mathcal{M}^c, \mu)$.
- $\mathbf{b}^*$) $|A|^{d+1}(\|B\|^{2d} + 1) \in L^{\text{loc}}(\mathcal{M}^c, \mu)$.

Then, for any $x_0 \in \mathbb{R}^d$, there is a solution X of (1.1) with $X_0 = x_0$.

The condition $f \in L^{\text{loc}}(\mathcal{M}^c, \mu)$ means that f is integrable with respect to μ over every compact subset K of $\mathcal{M}^c$ (and not of $[0,+\infty) \times \mathbb{R}^d$).
The proof follows the same steps as the proof of Theorem 3.2 in [3] (cf. [20]) and, therefore, is omitted. The following statement is then a natural extension of Corollary 3.7.

Corollary 3.11 *Suppose the coefficients A and B satisfy the following conditions:*

- $\mathbf{a_1^*}$) $\mathcal{M} \subseteq \mathcal{N}$.
- $\mathbf{c}^*$) *There exists $p > 1$ such that $|A|^{p(d+1)} \in L^{\text{loc}}(\mathcal{M}^c, \mu)$ and $\|B\|^{2q(d+1)} \in L^{\text{loc}}(\mathcal{M}^c, \mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

Then, for an arbitrary $x_0 \in \mathbb{R}^d$, there exists a solution X of the equation (1.1) with $X_0 = x_0$.

4 Time-Independent SDEs

Now suppose that the coefficients A and B do not depend on the time parameter t. In this case equation (1.1) becomes

$$X_t = x_0 + \int_0^t B(X_s)\, dW_s + \int_0^t A(X_s) ds, \quad t \geq 0, \tag{4.1}$$

where $x_0 \in \mathbb{R}^d$ and W is a d-dimensional Wiener process with $W_0 = 0$.
Let

$$d\bar{\mu}(y) = [\det \sigma(y)]^{-1}\, dy \tag{4.2}$$

and for any $m \in \mathbb{N}$, define

$$\bar{\tau}_m(X) := \inf\{t \geq 0 : |X_t| \geq m \text{ or } \int_0^t \text{trace}\, \sigma(X_s) ds \geq m \text{ or } \int_0^t |A(X_s)| ds \geq m\}.$$

First, we have the following version of Krylov's estimates that can be proven in the same way as Lemma 2.4.

Lemma 4.1 *Suppose X is a solution of SDE (4.1) and $f : \mathbb{R}^d \to [0, +\infty)$ is a nonnegative measurable function. Then there exists a constant C which depends on t, m, and d only such that the following inequality holds:*

$$\mathbf{E}\left[\int_0^{t\wedge\bar{\tau}_m(X)} f(X_s)\ [\det \sigma(X_s)]^{\frac{1}{d}}\ ds\right] \leq C\Big(\int_{U_m} f^d(y)\, dy\Big)^{\frac{1}{d}}.$$

The following theorem is then the analog of the existence Theorem 3.1 in the time-dependent case.

Theorem 4.2 *Suppose the following conditions are satisfied:*

$\bar{\mathbf{a}}_1$) $(\det B \cdot B^*)^{-1} \in L^{\text{loc}}(\mathbb{R}^d)$.

$\bar{\mathbf{a}}_2$) $\|B\|^{2d} \in L^{\text{loc}}(\mathbb{R}^d, \bar{\mu})$.

$\bar{\mathbf{b}}$) $|A|^d(\|B\|^{2d} + 1) \in L^{\text{loc}}(\mathbb{R}^d, \bar{\mu})$.

Then, for an arbitrary $x_0 \in \mathbb{R}^d$, there exists a solution X of the equation (4.1) with $X_0 = x_0$.

Corollary 4.3 *Suppose the coefficients A and B satisfy the following conditions:*

$\bar{\mathbf{a}}_1$) $(\det B \cdot B^*)^{-1} \in L^{\text{loc}}(\mathbb{R}^d)$.

$\bar{\mathbf{c}}$) *There exists* $p > 1$ *such that* $|A|^{pd} \in L^{\text{loc}}(\mathbb{R}^d, \bar{\mu})$
and $\|B\|^{2qd} \in L^{\text{loc}}(\mathbb{R}^d, \bar{\mu})$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

Then, for an arbitrary $x_0 \in \mathbb{R}^d$, *there exists a solution* X *of the equation (4.1) with* $X_0 = x_0$.

It is easy to see that in the one-dimensional case ($d = 1$) the condition $\bar{\mathbf{a}}_\mathbf{2}$) vanishes. The conditions of Theorem 4.2 (and of Corollary 4.3, correspondigly) become those found by H.J. Engelbert and W. Schmidt in [4]. If $A = 0$ then we have just one existence condition $\bar{\mathbf{a}}_\mathbf{1}$) that coincides with the existence condition found also by H. J. Engelbert and W. Schmidt in [5]. In fact, they proved that this is also a necessary condition for the existence of a non-trivial (not equal to a constant) solution of equation (4.1) for any initial value $x_0 \in \mathbb{R}$. In the case of one-dimensional SDEs with unit diffusion coefficient, we have the expected condition of local integrability of A.

We also can easily formulate the corresponding analogs of Theorem 3.10 and Corollory 3.11 in time-independent case.

Define

$$\bar{\mathcal{N}} = \{x \in \mathbb{R}^d : \ B(x) = 0 \quad \text{and} \quad A(x) = 0\}$$

and

$$\bar{\mathcal{M}} = \{x \in \mathbb{R}^d : \ \bar{\mu}(\bar{S}_\delta(x)) = \infty \quad \text{for all} \quad \delta > 0\},$$

where $\bar{S}_\delta(x)$ denotes the ball with center x and radius δ in $\mathbb{R}^d$.

Theorem 4.4 *Suppose the following conditions are satisfied:*

$\bar{\mathbf{a}}^*_\mathbf{1}$) $\bar{\mathcal{M}} \subseteq \bar{\mathcal{N}}$.

$\bar{\mathbf{a}}^*_\mathbf{2}$) $\|B\|^{2d} \in L^{\text{loc}}(\bar{\mathcal{M}}^c, \bar{\mu})$.

$\bar{\mathbf{b}}^*$) $|A|^d(\|B\|^{2d} + 1) \in L^{\text{loc}}(\bar{\mathcal{M}}^c, \bar{\mu})$.

Then, for any $x_0 \in \mathbb{R}^d$, *there is a solution* X *of (4.1) with* $X_0 = x_0$.

Corollary 4.5 *Suppose the coefficients* A *and* B *satisfy the following conditions:*

$\bar{\mathbf{a}}^*_\mathbf{1}$) $\bar{\mathcal{M}} \subseteq \bar{\mathcal{N}}$.

$\bar{\mathbf{c}}^*$) *There exists* $p > 1$ *such that* $|A|^{pd} \in L^{\text{loc}}(\bar{\mathcal{M}}^c, \bar{\mu})$
and $\|B\|^{2qd} \in L^{\text{loc}}(\bar{\mathcal{M}}^c, \bar{\mu})$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

Then, for an arbitrary $x_0 \in \mathbb{R}^d$, *there exists a solution* X *of the equation (4.1) with* $X_0 = x_0$.

Acknowledgement The authors would like to thank Prof. N.V. Krylov for his comments and the anonymous referee for the valuable remarks and suggestions which were incorporated in the final version and undoubtfully contributed to the improvement of the paper.

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