

# Multidimensional SDEs with Unbounded Drift

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November 24, 2003

## 1 Introduction

In this note we consider a stochastic equation of the form

$$X_t = x + \int_0^t B(s, X_s) dW_s + \int_0^t A(s, X_s) ds, \quad t \geq 0, \quad (1.1)$$

where  $B : [0, +\infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^{d^2}$ ,  $A : [0, +\infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  are Borel measurable matrix and vector functions with  $d \geq 1$ ,  $W$  is a  $d$ -dimensional Wiener process and  $x_0 \in \mathbb{R}^d$  is an arbitrary initial vector.

It is well-known that if the coefficients  $A$  and  $B$  satisfy the assumption of at most linear growth then the solution of the equation, if it exists, is nonexploding, i.e., it exists in  $\mathbb{R}^d$  for all  $t \geq 0$  (cf. [4], Theorem 6.4.2). In the general case the solution exists only in the sense that it may explode, i.e., on a finite time interval it may leave every compact subset of  $\mathbb{R}^d$ . Here we study solutions of Eq. (1) in this more general context.

The aim of this note is to give sufficient conditions for the existence of solutions to Eq. (1.1). A.V. Skorohod [13] was the first who investigated weak solutions of stochastic differential equations with continuous coefficients. N.V. Krylov [6] proved the existence of weak solutions of stochastic equations for only measurable coefficients using his well-known estimates for stochastic integrals of diffusion processes. Together with the boundedness of the drift coefficient, he only assumed that

$$c|z|^2 \leq (B(t, x)z, z), \quad \|B(t, x)\|^2 \leq C, \quad z \in \mathbb{R}^d,$$

for some constants  $0 < c < C$  not depending on  $(t, x) \in [0, +\infty) \times \mathbb{R}^d$  where  $(\cdot, \cdot)$  denotes the Euclidean scalar product,  $|\cdot|$  stands for Euclidean norm of vector  $z$ , and

$$\|B(t, x)\|^2 = \sum_{i,j=1}^d B_{ij}^2(t, x).$$

Using methods of nonstandard analysis, this result was generalized to certain degenerate diffusion matrices by S.A. Kosciuk [5].

The case of one-dimensional homogeneous equations (i.e., equations with time-independent coefficients) was treated by H. J. Engelbert and W. Schmidt [3]. In particular, there was shown that the local integrability of  $B^{-2}$  is necessary and sufficient for the existence of nontrivial solutions with an arbitrary initial value.

A more general case with time-dependent coefficients but still with one-dimensional state space was investigated by V. Kurenok [7], A. Rozkosz and L. Słomiński [9] and T. Senf [12]. T. Senf [12] was able to prove that the local integrability of  $B^2$  and  $B^{-2}$  ensures the existence of a solution for every initial value and, moreover, the solution does not explode if only, for every  $N \geq 1$ , there exists a nonnegative function  $\bar{B}_N$  finite on a set of positive Lebesgue measure such that  $B^2(t, x) \leq \bar{B}_N(x)$  for every  $t \in [0, N]$  and  $x \in \mathbb{R}$ .

Another far-reaching generalization was given by A. Rozkosz and L. Słomiński [10], [11] for multi-dimensional stochastic equations with time-independent and also with time-dependent diffusion and drift coefficients satisfying, additionally, the usual linear growth condition. In particular, their results also include those of A.S. Kosciuk [5].

In the paper of H.J.Engelbert and V.P.Kurenok [2] the sufficient existence conditions for the SDEs (1.1) without drift were found those generalize the known results for the one-dimensional case and seem to be very close to necessary conditions. Our motivation was to extend these conditions to the general case with drift.

By discussing results about SDEs with unbounded drift one should mention the results of N.M.Portenko [8] who proved the existence of solutions for the equation (1.1) with smooth diffusion coefficient and the drift satisfying some local integrability conditions. In order to prove an existence theorem for possibly most general drift, he was required to put a much stronger condition on the diffusion coefficient. As an essential tool for proving his results, he used his own estimate (similar to Krylov's estimates) for stochastic integrals of SDEs solutions with locally integrable drift.

In the time-dependent or multidimensional cases, main tools remain Krylov's estimates. The proof of our main result is similar to that from [2].

## 2 Existence of Solutions

We first begin with the definition of a solution of (1.1). For any stochastic process  $Y$  and  $a \in \mathbb{R}_+$  we set

$$\tau_\infty(Y) = \inf\{t \geq 0 : |Y_t| = \infty\} \quad \text{and} \quad \tau_a(Y) = \inf\{t \geq 0 : |Y_t| \geq a\}.$$

A stochastic process  $(X, \mathbb{F})$ , defined on a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  with filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  is called a solution of (1.1) with initial state  $x_0 \in \mathbb{R}^d$  if there exists a  $d$ -dimensional Wiener process  $W = (W_t)_{t \geq 0}$  with respect to the filtration  $\mathbb{F}$  such that  $W_0 = 0$  and

$$X_t = x_0 + \int_0^t B(s, X_s) dW_s + \int_0^t A(s, X_s) ds \quad \text{on} \quad \{t < \tau_\infty(X)\} \quad \mathbf{P}\text{-a.s.} \quad (2.1)$$

for all  $t \geq 0$ .  $\tau_\infty(X)$  is called the *explosion time* of  $X$ . Solutions of this type are sometimes called *weak* solutions. Obviously, Eq. (2.1) is equivalent to

$$X_{t \wedge \tau_m(X)} = x_0 + \int_0^{t \wedge \tau_m(X)} B(s, X_s) dW_s + \int_0^{t \wedge \tau_m(X)} A(s, X_s) ds \quad \mathbf{P}\text{-a.s.}, \quad (2.2)$$

where  $t \geq 0, m \in \mathbb{N}$ .

Let  $f$  be a measurable function on  $[0, +\infty) \times \mathbb{R}^d$ . We use the notation  $f \in L^{\text{loc}}([0, +\infty) \times \mathbb{R}^d)$  if  $f$  is locally integrable, i.e., integrable with respect to the Lebesgue measure on every compact subset of  $[0, +\infty) \times \mathbb{R}^d$ . We define  $\sigma$  by

$$\sigma(t, x) = B(t, x) \circ B^*(t, x), \quad (t, x) \in [0, +\infty) \times \mathbb{R}^d,$$

where  $B^*$  denotes the transpose of a matrix  $B$ . Clearly,  $\sigma(t, x)$  is a symmetric and nonnegative definite matrix. Let

$$d\mu(s, y) = [\det \sigma(t, y)]^{-1} dy ds$$

where  $0^{-1} = +\infty$ . Similarly, the notation  $f \in L^{\text{loc}}([0, +\infty) \times \mathbb{R}^d, \mu)$  stands for the local integrability of  $f$  with respect to the measure  $\mu$  on  $[0, +\infty) \times \mathbb{R}^d$ .

The following lemma is a version of Krylov's estimates for stochastic integrals (also see H.J. Engelbert and V.P. Kurenok [2], A. Rozkosz and L. Słomiński [9], [11]) and is a crucial tool in the proof of our main result. For all  $m \in \mathbb{N}$  we define the ball  $U_m = \{x \in \mathbb{R}^d : |x| \leq m\}$  around the origin with radius  $m$  and let  $K > 0$  be a constant.

**Lemma 2.1** *Let  $X_t$  be a solution of SDE (1.1) with locally bounded drift  $A$  (such that  $|A(s, x)| \leq K$  for all  $0 \leq s \leq t, x \in U_m$ ) and  $f : [0, +\infty) \times \mathbb{R}^d \rightarrow [0, +\infty)$  be a nonnegative measurable function. Then there exists a constant  $C$  which depends only on  $t, m, d$ , and  $K$  such that the following inequality holds:*

$$\mathbf{E} \int_0^{t \wedge \tau_m(X)} f(s, X_s) [\det \sigma(s, X_s)]^{\frac{1}{d+1}} ds \leq C \left( \int_{[0, t] \times U_m} f^{d+1}(s, y) dy ds \right)^{\frac{1}{d+1}}.$$

We need the following three conditions:

- a<sub>1</sub>)**  $(\det B \circ B^*)^{-1} \in L^{\text{loc}}([0, +\infty) \times \mathbb{R}^d)$ .
- a<sub>2</sub>)**  $\|B\|^{2(d+1)} \in L^{\text{loc}}([0, +\infty) \times \mathbb{R}^d, \mu)$ .
- b)**  $A$  is locally bounded (bounded on every compact subset of  $[0, +\infty) \times \mathbb{R}^d$ ).

Obviously, we have

$$\|B\|^2 := \sum_{i,j=1}^d B_{ij}^2 = \text{trace } \sigma.$$

In [2] the existence of a (possibly, exploding) solution to Eq. (1.1) was proven under the assumptions  $a_1)$  and  $a_2)$ . The next theorem generalizes this result to the case of SDE's with diffusion and drift coefficients.

**Theorem 2.2** *Suppose that the conditions **a<sub>1</sub>)**, **a<sub>2</sub>)** and **b)** are satisfied. Then, for an arbitrary  $x_0 \in \mathbb{R}^d$ , there exists a solution  $X$  of Eq. (1.1) with  $X_0 = x_0$ .*

*Proof.* We are going just to sketch the main idea of the proof. The details of the proof are the subject of a forthcoming paper.

Because  $\sigma(t, x)$  is symmetric and nonnegative, we can find orthogonal matrices  $U(t, x)$ , which can be chosen measurable in  $(t, x)$ , such that

$$\Lambda(t, x) = U^*(t, x) \circ \sigma(t, x) \circ U(t, x), \quad (t, x) \in [0, +\infty) \times \mathbb{R}^d,$$

are of diagonal form with nonnegative diagonal elements  $\lambda_i(t, x)$ ,  $i = 1, 2, \dots, d$ . Equivalently,  $\sigma$  has the representation

$$\sigma(t, x) = U(t, x) \circ \Lambda(t, x) \circ U^*(t, x), \quad (t, x) \in [0, +\infty) \times \mathbb{R}^d.$$

For  $n \in \mathbb{N}$  consider the diagonal matrix function  $\Lambda_n$  with diagonal elements  $\lambda_i^{(n)} = (\lambda_i \vee \frac{1}{n}) \wedge n$ ,  $i = 1, 2, \dots, d$ , where  $\vee$  and  $\wedge$  stand for maximum and minimum, respectively. Let

$$B_n = \sqrt{2}U \circ \Lambda_n^{\frac{1}{2}} \circ U^*, \sigma^{(n)} = \frac{1}{2}B_n \circ B_n^* \quad \text{and} \quad A_i^n = (A_i \vee -n) \wedge n \quad i = 1, 2, \dots, d.$$

In view of  $\|B_n\|^2 = \text{trace } \sigma^{(n)}$  and

$$\max_{i,j=1,\dots,d} \sigma_{ij} \leq \max_{i=1,\dots,d} \lambda_i \leq \text{trace } \sigma \leq d \max_{i=1,\dots,d} \sigma_{ii}$$

we get

$$\|B_n\|^2 \leq dn, \quad n \in \mathbb{N}.$$

Also we get

$$|A^n| \leq \sqrt{dn}, n \in \mathbb{N}.$$

Furthermore, for every  $z \in \mathbb{R}^d$

$$(B_n z, z) = (U \Lambda_n^{\frac{1}{2}} U^* z, z) = (\Lambda_n^{\frac{1}{2}} U^* z, U^* z) \geq n^{-\frac{1}{2}} |U^* z|^2 = n^{-\frac{1}{2}} |z|^2.$$

Therefore, the coefficients  $B_n$  and  $A^n$  satisfy the assumptions of Krylov's theorem (cf. [6], Theorem 2.6.1) and hence there exist probability spaces  $(\Omega^n, \mathcal{F}^n, \mathbf{P}^n)$  with filtrations  $\mathbb{F}^n = (\mathcal{F}_t^n)_{t \geq 0}$  and processes  $(X^n, \mathbb{F}^n)$  and  $(W^n, \mathbb{F}^n)$  such that  $(W^n, \mathbb{F}^n)$  are Wiener processes and  $(X^n, W^n)$  satisfy Eq. (1.1) with initial value  $X_0^n = x_0$ .

The next step in proving the Theorem is to show that the sequence of processes  $(X^n_{\cdot \wedge \tau_m(X^n)})_{n \in \mathbb{N}}$ , for arbitrary but fixed  $m$ , is tight in sense of weak convergence of stochastic processes with continuous trajectories. Due to a theorem of D. Aldous [1], the sequence  $(X^n), n = 1, 2, \dots$  is tight if for every sequence  $(\tau^n)$  of  $\mathbb{F}^n$ -stopping times and every sequence  $(\delta_n)$  of real numbers such that  $\delta_n \downarrow 0$  and all  $\varepsilon > 0$  it follows

$$\lim_{n \rightarrow \infty} \mathbf{P}^n(|X^n_{t \wedge (\tau^n + \delta_n) \wedge \tau_m(X^n)} - X^n_{t \wedge \tau^n \wedge \tau_m(X^n)}| > \varepsilon) = 0.$$

This fact can be verified by using the Chebyshev inequality, Krylov's estimates and the conditions of the Theorem following similar steps as in [2]. Hence we have a sequence of measures  $P^{X^n}, n \geq 1$  that converges to a measure  $Q$  defined on the space of continuous (possibly, exploding) trajectories. Next we can construct a process  $X$  with the distribution that coincides with  $Q$ . The remaining step is then to show that the process  $X$  satisfies the original equation (1.1). This can be done by means of weak convergence of stochastic processes and Krylov's estimates. We omit the details.

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