ON A MODEL FOR THE TERM STRUCTURE OF INTEREST RATE PROCESSES OF STABLE TYPE

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Abstract

Let $M(t), t \geq 0$, be an arbitrary one-dimensional symmetric stable process. As a model for the term structure of interest rate processes we consider $r(t) = \mathcal{G}(t, M \circ T(t))$ or as special case $r(t) = F\Big(f(t) + g(t)M \circ T(t)\Big)$ for some functions $\mathcal{G}, F, T, f$ and g. We show that this model includes in particular some models which can be described by the Ito stochastic differential equations driven by the stable process M. It generalizes also the known Schmidt's model which is the special case of our model. Moreover, we construct also a sequence of simple processes (random walks) obtained as the sums of i.i.d random variables which belong to the domain of attraction of the corresponding stable distribution. It is proved that this random walk models converge in law to the interest rate processes $r(t)$.

Key Words: Short rate, stochastic differential equations, stable processes, random walk

1 Introduction

The interest rate process describes the profibility of some financial instrument, such as stock, bond or option. Hence, if the price change is given by the sequence $X = (X_n)_{n\geq 0}$ then in the simplest case the interest rate process has the form

$$r_n = \frac{\triangle X_n}{X_{n-1}}, \text{ where } \triangle X_n = X_n - X_{n-1}.$$

The role of interest rate can better be understood by writing the sequence (X_n) in the form $X_n = X_0 \exp(H_n)$ with $H_n = \sum_{i=1}^n h_i$. It is easily verified that $X_n = X_0 \prod_{i=1}^n (1+r_i)$, where $r_i = \ln(1+h_i)$. In other words, if X_0 denotes the initial value of the financial sequence X, then the nth value is equal to the product of X_0 and n return-values. Generating the above relation for continuous time we obtain

$$dX_t = r_t X_t dt, \ t \geq 0.$$

But in practice another form of dependence of the processes (r_t) and (X_t) is often used, namely the equation

$$dr_t = \sigma(t, r_t) dX_t, \ t \geq 0.$$

The process (r_t) is then called a short interest rate process. If the process X is a semimartingale then this equation can be written as

$$r_t = r_0 + \int_0^t \sigma(s, r_{s-}) dX_s,$$

where the integral is the stochastic integral with respect to a semimartingale and $r_{s-} = \lim_{t\uparrow s} r_t$ for all $s \geq 0$ ([13], [16]). The interest rate processes of diffusion type

$$dr_t = a(t, r_t)dt + b(t, r_t)dW_t, \ t \geq 0, \tag{1.1}$$

play an important role in the case of continuous semimartingales. Here $a, b : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ are measurable functions and W is an one-dimensional brownian motion with $W_0 = 0$.
W. M. Schmidt considered in [15] a short interest rate model of the form

$$r_t = F\Big(f(t) + g(t) W^{\star}_{T(t)}\Big), \tag{1.2}$$

where $W^\star$ is some brownian motion, $T(t), F(x)$ are nonnegative, continuous, strictly increasing functions in $t \geq 0$ and $x \in \mathbb{R}$, respectively, and $f(t), g(t)$ are continuous functions with $g(t) > 0,\ t \geq 0$. The Schmidt's model (1.2) includes some known models of interest rate processes (r_t) being solutions of the stochastic equation (1.1) which can be obtained from (1.2) by an appropriate choice of "driving" components F, f, g and T. Moreover, some random walk models approximating the continuous model (1.2) were discribed in [15].

In this paper, first of all, we consider the following generalization of the stochastic differential equation (1.1)

$$r_t = r_0 + \int_0^t a(s, r_s)ds + \int_0^t b(s, r_s)dM_s,\ t \geq 0, \tag{1.3}$$

as a model for interest rate processes (r_t). We suppose in (1.3) that $(M_t)_{t\geq 0}$ is a symmetric stable process of index $\alpha \in (0, 2]$ and the integral with respect to M is the stochastic integral of Ito type with respect to stable processes ([5], [14]). The process (r_t) has the property of "heavy tails" distributions similar to the driving process M ([7]) that is verified by many real financial dates for short rates ([16], Chap. 4).
Generalizing the Schmidt model (1.2) we introduce in section 2 the following model for (r_t)

$$r_t = F\Big(f(t) + g(t)M^\star_{T(t)}\Big), \tag{1.4}$$

where $M^\star$ is some symmetric stable process of index α and the functions F, f, g and T satisfy the same assumptions as in model (1.2). By using properties of Ito integrals with respect to stable processes in section 3 we consider some types of equation (1.3) the solutions of which can be represented in the form (1.4).

In section 4 we also describe a sequence of simple interest rate processes $(r^n_t), n \geq 1$, which converges in law to the process (r_t). The processes r^n are represented as appropriate functionals of finite sums of i.i.d. random variables which belong to the domain of attraction of the corresponding symmetric stable process. The described approximation procedure which is of independent interest consists in the approximation of arbitrary symmetric stable processes through some more elementary stable random walk processes. We prove the convergence of the distributions of simple random walk processes on the space of their trajectories. This our result generalizes the results of Gorenflo and Mainardi ([9], [10]) who considered random walk approximation procedures for stable processes and it was proved there the convergence to the one-dimensional distributions of stable processes, i.e., for every fixed time-parameter $t > 0$.

2 A general model for interest rate processes

Let $\mathbb{D}_{[0,\infty)}(\mathbb{R}) = \{x : [0,\infty) \to \mathbb{R}\}$ be the space of right continuous functions with finite left limits which is often called the space of cadlag functions.

Definition 2.1. *A process $(M_t)_{t\geq 0}$ with $M_0 = 0$ and with trajectories in $\mathbb{D}_{[0,\infty}(\mathbb{R})$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is called a stable process of index $\alpha \in (0, 2]$ if it is stochastic continuous and*

$$\mathbf{E}\Big(e^{i\lambda(M_t-M_s)} \mid \mathcal{F}_s^M\Big) = \exp\Big((t-s)\Big[i\lambda\gamma + C_1 \int\limits_{-\infty}^{0} (e^{i\lambda u} - 1 - \frac{i\lambda u}{1+u^2})\nu(du) +$$

$$C_2 \int\limits_{0}^{\infty} (e^{i\lambda u} - 1 - \frac{i\lambda u}{1+u^2})\nu(du)\Big]\Big), \qquad (2.1)$$

where $\mathcal{F}_s^M = \sigma(M_u : 0 \leq u \leq s)$ for all $0 \leq s \leq t < \infty; \gamma, C_1$ and C_2 are some real constants and

$$\nu(du) = \frac{du}{\mid u \mid^{1+\alpha}}.$$

The measure ν is called the Lévy measure of the process M. The equality (2.1) can be also written in the form (cf. [17], Chap. 5)

$$\mathbf{E}\Big(\exp i\lambda(M_t - M_s) \mid \mathcal{F}_s^M\Big) = \exp\Big((t-s)[i\delta\lambda - c \mid \lambda \mid^\alpha (1 - iq\frac{\lambda}{\mid \lambda \mid}k(\lambda,\alpha)]\Big)$$

for all $t > s \geq 0,\ \lambda \in \mathbb{R}$, where

$$k(\lambda,\alpha) = \begin{cases} \tan \pi\alpha/2 & \text{if } \alpha \neq 1, \\ (2/\pi)\ln \mid \lambda \mid & \text{if } \alpha = 1. \end{cases}$$

Here $c > 0,\ \delta \in \mathbb{R},\ q \in [-1, 1]$ are the scale, location and skewness parameter, respectively [17]. The parameter α is called the stability index of the process M. In the special case of $\delta = 0$ and $q = 0$ (respectively, $C_1 = C_2$ and

$\gamma = 0$) the process M is called the symmetric stable process of the index α. Using the property of stable distributions it can be seen that for all $\tau > 0$ the process $(M_{\tau t})$ is again a stable process with the same parameters α and q but with other parameters δ and c. We can always reduce the process M to another stable process with $c = 1$ which is called the standard process. For $\alpha = 2$ in the symmetric standard case we have the brownian motion, which is the only stable process with continuous trajectories. For $\alpha < 2$ the trajectories of M are purely discontinuous: For $1 < \alpha < 2$ the process is a martingale and for $0 < \alpha \leq 1$ it is a process with locally bounded variation [1].

It is known that for all $\alpha \in (0, 2]$ the stable process M is a semimartingale. It means that a stochastic integral with respect to this process can be defined using the general semimartingale approach ([13], [16]). But in the case $\alpha < 2$ this approach gives only general conditions to guarantee the existence of the stochastic integral. If $\alpha = 2$ then the stochastic integral is defined for a maximally wide class of integrands due to L_2-isometrie (cf. [12]). It is remarkable that for $0 < \alpha < 2$ it is also possible to apply the basic Ito concept to the definition of the integral. Since the random stable variable M_t of the index α doesn't belong to the L_α-space of random variables for all $t > 0$, there is not an L_α-isometrie between the space of integrands and the space of stochastic integrals as in the case $\alpha = 2$. But M_t belongs to the weak L_α-space or Lorenz-Marzinkewicz space [7]. The semi-norm on this space satisfies a two-side estimate via L_α-norm due to the properties of stable distributions. The corresponding conditions on integrands become then the necessary and sufficient conditions for the existence of stochastic integrals ([5], [14]). This approach also implies other usefull properties of stochastic integrals. One important property is that every stochastic integral with respect to the symmetric stable process can be reduced by an appropriate time change to another stable process with the same index.

The Ito approach is also the main tool for investigation of our general model of interest rate processes. This model has the form

$$r_t = \mathcal{G}\Big(t, M^\star_{T(t)}\Big),\ t \geq 0, \tag{2.2}$$

where the process $M^\star$ is a standard symmetric stable process of the index α, $T(t)$ with $T(0) = 0$ is a continuous strictly increasing function and $\mathcal{G}(t, x)$ is also a continuous in both variables function. The model (1.4) is a special case of the equation (2.2). As in the Schmidt's model ($\alpha = 2$) this special case allows us to get more information about the process (r_t). It can easily be verified that the process $Y_t = f(t) + g(t)M^{\star}_{T(t)}$ is again a stable process with the same stability index α and skewness parameter q, but with another location parameter δ and scale parameter c. If $S^t_{(\delta,c,q,\alpha)}(x), x \in \mathbb{R}$, is the distribution function of the stable variable M_t, therefore $S^t_{(0,1,0,\alpha)}$ is that of $M^{\star}_t$, then the distribution function of r_t determined by (1.4) can be represented in the form $S^t_{(\delta,c,q,\alpha)}(x) \circ F^{-1}$, where F^{-1} is the inverse function to F.

3 Solutions of stochastic equations which can be expressed by the general model

A) Let us consider the following stochastic differential equation for the process (r_t)

$$dr_t = \Big(\theta(t) - \beta(t)r_t\Big)dt + \gamma(t)dM_t,\ t \geq 0. \tag{3.1}$$

We call it a generalized Hull-White model because in the case of $\alpha = 2$ it is known as Hull-White model ([11], [18]). We assume that the functions θ, β and γ are nonrandom.

Lemma 3.1. *The solution of equation (3.1) can be expressed in the form (1.4).*

Proof. We put

$$r_t = g(t)\Big[r_0 + \int\limits_0^t \frac{\theta(s)}{g(s)}ds + \int\limits_0^t \frac{\gamma(s)}{g(s)}dM_s\Big], \tag{3.2}$$

where M is any standard symmetric stable process of the index α, $r_0 \in \mathbb{R}$ is an arbitrary initial value and $g(t) = \exp\Big(-\int\limits_0^t \beta(s)ds\Big)$.

Using integration by parts (Ito-formula) for semimartingales ([13], p. 99) one can easily verify that the process (3.2) is a solution of the equation (3.1).

Let us denote

$$T_t = \int\limits_0^t \mid \frac{\gamma(s)}{g(s)} \mid^{\alpha} ds,\ t \geq 0.$$

It is known from the properties of stable integrals that the function (T_t) is a so-called "inner clock" for the stochastic integral in (3.2). Moreover, if

$$|\frac{\gamma}{g}|^{\alpha} \in L^{loc} \text{ and } T_{\infty} = \lim_{t\to\infty} T(t) = \infty \tag{3.3}$$

then there exists a stable process $M^{\star}$ of the same index α so that

$$\int_0^t \frac{\gamma(s)}{g(s)} dM_s = M^{\star}_{T(t)}$$

(cf. [5], [14]).

Then it follows from (3.2) that

$$r_t = f(t) + g(t)M^{\star}_{T(t)},$$

where

$$f(t) = g(t)\Big[r_0 + \int_0^t \frac{\theta(s)}{g(s)} ds\Big]. \qquad \square$$

Remark 3.1. In (3.3) the condition $T_{\infty} = \infty$ is not necessary. In the general case the stochastic integral in (3.2) can be represented as another stable process stopped in the stopping time T_{∞} (cf. [5]).

B) Generalizing the case $\alpha = 2$ we consider the SDE

$$dr_t = r_t\Big(\theta(t) - \beta(t)\ln r_t\Big)dt + \gamma(t) r_t dM_t,\ t \geq 0, \tag{3.4}$$

which we call a generalized Black-Karasinski model (cf. e.g. [2]).

Lemma 3.2. *The model (3.4) can be reduced to the form (1.4).*

Proof. Obviously, we can rewrite the equation (3.4) in the equivalent form

$$d\ln r_t = \Big[\theta(t) - \beta(t)\ln r_t\Big]dt + \gamma(t)dM_t,$$

which can be solved analogously to the equation (3.1). Its solution has the form

$$r_t = F\Big(f(t) + g(t)M^{\star}_{T(t)}\Big),$$

where the functions $g(t)$ and $T(t)$ are defined in Lemma 3.1 and

$$f(t) = g(t)\Big[r_0 + \int\limits_0^t \frac{\theta(s)}{g(s)} ds\Big],$$

$$F(x) = \exp(x). \qquad \Box$$

C) In an analogous way we can treat the following generalization of model (3.1)

$$dr_t = \Big[\theta(t) - r_t\Big]dt + \gamma(t)dM_t^1,$$

where the coefficients $\theta(t)$ and $\gamma(t)$ are the solutions of the SDE's of the same type

$$d\theta(t) = \Big[\theta - \theta(t)\Big]dt + \beta_1(t)dM_t^2,$$

$$d\gamma(t) = \Big[\gamma - \gamma(t)\Big]dt + \beta_2(t)dM_t^3$$

and M^1, M^2, M^3 are independent stable processes of the same index (cf. [3], and [16], Chap. 3, sec. 4).

D) We call the model given by

$$dr_t = \theta(t)r_t dt + \gamma(t)r_t dM_t$$

a generalized Dothan model (see [4]). Obviously, the solution (r_t) has the form (1.4) with

$$F(x) = \exp(x), \quad f(t) = \int\limits_0^t \theta(s)ds,$$

$$g(t) \equiv 1, \quad T(t) = \int\limits_0^t \mid \gamma \mid^p ds.$$

Remark 3.2. Analogously to the case $\alpha = 2$ we can also introduce generalized Dothan model as a model for geometric stable motion which is a generalization of geometric brownian motion.

4 The convergence of distributions of random walk processes to the distributions of stable processes

In theory of random processes as well as in its many applications it is important to find the distributions of various functionals of some basic stochastic processes. So, the formula (1.4) or the general formula (2.2) are examples of such functionals of symmetric stable processes. It is also important to know the distributions of the interest rate processes in the corresponding financial problems. Thus, the contingent claims $C(r_t)_{0\leq t\leq \tau}$ on the bond markets, for example,depending on (r_t) are usually expressed in the form

$$C(r_t) = \mathbf{E_Q}\Big(\exp\{-\int_t^\tau r_u du\}C(r_\tau) \mid \mathcal{F}_t\Big),$$

where τ is the bond maturity time, $(\mathcal{F}_t)$ is a filtration and Q is an appropriate martingale measure (cf. [15]). To calculate $C(r_t)$ we need to know the distribution of the process (r_t). For the pricing of the contingent claims it would be usefull to have a sequence of more simple processes (r_t^n), $n \geq 1$, which converges in a suitable sense to the process (r_t). It should allow us to use the corresponding sequence

$$\mathbf{E_Q}\Big(\exp\{-\int_t^\tau r_u^n du\}C(r_\tau^n) \mid \mathcal{F}_t\Big),\ n \geq 1,$$

for approximating $C(r_t)$.

In this section we describe one such sequence of processes (r_t^n) which approximates the process (r_t) of the form (2.2). For every fixed n the process (r_t^n) will be determined as a sum consisting of only n independent identically distributed random variables. Furthermore, it holds $r^n \to^{\mathcal{D}} r$ for $n \to \infty$ that means the convergence of distributions of the processes r^n to the distribution of the process r.

First of all we note that the characteristic function (2.1) of the symmetric stable process can also be written in the form

$$\mathbf{E}e^{i\lambda M_t} = \exp\Big\{t\int_{-\infty}^{\infty}(e^{i\lambda u} - 1 - \frac{i\lambda u}{1+u^2})\frac{1+u^2}{u^2}dG(u)\Big\},$$

where

$$dG(u) = \frac{u^2}{1+u^2}\nu(du).$$

Obviously, the function $G(u)$ is then monotonous and bounded because the density of the measure dG with respect to the Lebesgue measure is integrable over $\mathbb{R}$.

We remind that a random variable ξ has a symmetric stable distribution of the index $\alpha \in (0,2]$ if its characteristic function has the following form

$$\psi_\alpha(\lambda) = \mathbf{E}e^{i\lambda\xi} = \exp\Big(\int\limits_{-\infty}^{\infty}(e^{i\lambda u} - 1 - \frac{i\lambda u}{1+u^2})\nu(du)\Big).$$

The function

$$U_\alpha(x) = \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{-i\lambda x}\psi_\alpha(\lambda)d\lambda,\ \ x \in \mathbb{R},$$

is then called the distribution function of the r. v. ξ.

Let $\Big(\xi_n\Big)_{n\geq 1}$ be a sequence of i.i.d. symmetric random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. For every $n \in \mathbb{N}$ and $\tau \in [0,\infty)$ consider the stochastic process $(M^n_{t,\tau}), t \in [0,\tau]$, defined as

$$M^n_{t,\tau} = \begin{cases} \sum_{i=1}^{k}\xi_i \text{ if } t \in [\frac{k-1}{n}\tau, \frac{k}{n}\tau), \\ \sum_{i=1}^{n}\xi_i \text{ if } t = \tau. \end{cases} \tag{4.1}$$

We call the process $M^n_{\cdot,\tau}$ a random walk process.

For every $n \in \mathbb{N}$ and $\tau \in [0,\infty)$ the process $M^n_{\cdot,\tau}$ is a cadlag process with independent increments, in particular, $M^n_{\cdot,\tau} \in D_{[0,\tau]}(\mathbb{R})$.

We are interested in conditions guaranteeing the $\mathcal{D}$- convergence of the processes $(M^n_{t,\tau})_{0\leq t\leq\tau}$ to a symmetric stable process $(M_t)_{0\leq t\leq\tau}$.

Set $S_n = \sum_{i=1}^{n}\xi_i,\ n \geq 1$, and let F be the common distribution function of the random variables $\xi_i, i \geq 1$.

Definition 4.1. *We say that the distribution function F belongs to the (normal) domain of attraction of the distribution U which is not concentrated in one point if there are constants $a_n > 0$ such that for all $x \in \mathbb{R}$ $P_{\frac{1}{a_n}S_n}(x) \to U(x)$ as $n \to \infty$ where $P_{\frac{1}{a_n}S_n}$ denotes the distribution function of $\frac{1}{a_n}S_n$.*

It is known that only the distributions $U_\alpha(x), x \in \mathbb{R}, \alpha \in (0,2]$, have a domain of attraction.

Define

$$\gamma_n = n \int_{-\infty}^{\infty} \frac{x}{1+x^2} dF_n(x),$$

$$G_n(x) = n \int_{-\infty}^{x} \frac{u^2}{1+u^2} dF_n(u),$$

where F_n denotes the distribution function of the random variables $a_n^{-1}\xi_i, i \geq 1$.

Definition 4.2. *For the monotonous bounded functions $G_n(x)$ and $G(x)$ we write $G_n(x) \Rightarrow G(x)$ for $n \to \infty$ to denote the convergence in all continuity points $x \in \mathbb{R}$.*

The following result is due to A. V. Skorohod and gives a sufficient condition for the $\mathcal{D}$-convergence of sums of i.i.d. random variables to a symmetric stable process.

Proposition 4.1. *(cf. Theorems 1, 2 in [8], Chap. 9, §6). If $\gamma_n \to 0$ and $G_n(x) \Rightarrow G(x)$ for $n \to \infty$ then for every $\tau \in [0, \infty)$ $(M^n_{t,\tau})_{0 \leq t \leq \tau} \to^{\mathcal{D}} (M_t)_{0 \leq t \leq \tau}$ for $n \to \infty$.*

Consider the following sequence of numbers $\big(p_k(\alpha)\big)_{k \in \mathbb{Z}}$ defined as

$$\begin{cases} p_0 = 1 - \mu, \\ p_{\mp k} = (-1)^{k+1} \mu d \binom{\alpha}{k}, \ k = 1, 2, ... \end{cases}$$

for $0 < \alpha < 1$ and $0 < \mu \leq \cos(\frac{\alpha\pi}{2})$, or

$$\begin{cases} p_0 = 1 + \mu\alpha(\cos\frac{\alpha\pi}{2})^{-1}, \\ p_{\mp 1} = -\mu(d\binom{\alpha}{2} + d), \\ p_{\mp k} = (-1)^k \mu d \binom{\alpha}{k+1}, \ k = 2, 3, ... \end{cases}$$

for $1 < \alpha \leq 2$ and $0 < \mu \leq \alpha^{-1}\cos(\frac{\alpha\pi}{2})$, or

$$\begin{cases} p_0 = 1 - \frac{2\mu}{\pi}, \\ p_{\mp k} = \mu\pi \mid k \mid (\mid k \mid + 1), \ k = 1, 2, ... \end{cases}$$

for $\alpha = 1$ and $0 < \mu \leq \frac{\pi}{2}$, where $d = \frac{\sin\frac{\pi}{2}\alpha}{\sin\pi\alpha}$.

It can easily be seen that $p_k(\alpha) \geq 0, \ k \in \mathbb{Z}$, and $\sum_{k \in \mathbb{Z}} p_k(\alpha) = 1$ for all $\alpha \in (0, 2]$ (cf. [10]), i.e., $\big(p_k(\alpha)\big)_{k \in \mathbb{Z}}$ defines a probability distribution. We note that it is also symmetric.

Assume

$$P(\xi_i = k) = p_k(\alpha),\ \ k \in \mathbb{Z} \tag{4.2}$$

for all $i \geq 1$. For the proof of the main statement of this section (Theorem 4.1) we need the following result about the distributions $\big(p_k(\alpha)\big)$ which immediately follows from Theorem 3.2 in [10].

Proposition 4.2. *For all $\alpha \in (0,2]$ the probability distribution F_α defined as*

$$F_\alpha(x) = \sum_{k \leq x} p_k(\alpha),\ \ x \in \mathbb{R},$$

belongs to the domain of attraction of the distribution U_α.

Theorem 4.1. *Suppose that the sequence of the processes $(M^n_{t,\tau}), t \in [0,\tau]$, is defined as in (4.1) and the corresponding i.i.d. symmetric random variables $\xi_i, i \geq 1$, have the distribution $\big(p_k(\alpha)\big)_{k\in\mathbb{Z}}$ as defined above. Then we have $(M^n_{t,\tau})_{0\leq t\leq\tau} \to^{\mathcal{D}} (M_t)_{0\leq t\leq\tau}$ for $n \to \infty$ where M is a symmetric stable process of the index $\alpha \in (0,2]$.*

Proof. Using the Proposition 4.1 it is sufficient to show

$$\gamma_n \to 0 \qquad \text{and} \qquad G_n(x) \Rightarrow G(x),\ n \to \infty.$$

From Proposition 4.2 follows that there exist constants $a_n > 0$ such that the sequence of distributions of random variables $a_n^{-1}\sum_{i=1}^n \xi_i$ converges as $n \to \infty$ to the distribution U_α where the random variables ξ_i are such that as in (4.2). In order to check the conditions of the Proposition 4.1 we remark that we have $F_n(x) = F_{n,\alpha}(x) := F_\alpha(a_n x)$. Let ϕ_α be the characteristic function of the distribution F_α. Obviously, the statement F_α *belongs to the domain of attraction of U_α* is equivalent to

$$\phi^n_\alpha(\lambda a_n^{-1}) \to \psi_\alpha(\lambda) \text{ by } n \to \infty \tag{4.3}$$

for every $\lambda \in \mathbb{R}$.

It is easy to prove that the relation (4.3) can be written in the form

$$n\Big[\phi_\alpha(\lambda a_n^{-1}) - 1\Big] \to \omega(\lambda), \lambda \in \mathbb{R} \text{ by } n \to \infty, \tag{4.4}$$

where $\psi_\alpha(\lambda) = e^{\omega(\lambda)}$ (cf. Theorem 1 in [6], Chap. XVII, §1).

As in the proof of the Theorem about the canonical representation of infinitly divisible characteristic functions ([6], Theorem 1, Chap. XVII, §2) it follows from (4.4)

$$nF_{n,\alpha}(dx) \Rightarrow \nu(dx) \text{ as } n \to \infty, \tag{4.5}$$

where the measure ν is defined in (2.1).

Taking into account the definitions of the functions $G_n(x)$ and $G(x)$ we obtain

$$G_n(x) \Rightarrow G(x) \text{ as } n \to \infty.$$

From (4.5) follows that the relation $\gamma_n \to 0$ as $n \to \infty$ is also satisfied that proves the theorem.

Theorem 4.1 can be extended to more general processes (r_t) of the form (1.4) or (2.2) in the following way. First of all we remind a well-known fact that for every $\tau \in (0, \infty)$ the Skorohod space $D_{[0,\tau]}(\mathbb{R})$ is a separable metric space according to the metric ρ_D defined as

$$\rho_D(x, y) = \inf_{\lambda \in \Lambda} \{\sup \mid x(t) - y(\lambda(t)) \mid + \sup_t \mid t - \lambda(t) \mid\}$$

where $x, y \in D_{[0,\tau]}(\mathbb{R})$ and $\Lambda = \{\lambda : [0, \tau] \to [0, \tau],\ \lambda$ are monotonous and continuous with $\lambda(0) = 0, \lambda(\tau) = \tau\}$.

Collorary 4.1. Set $r_t^n = \mathcal{G}(t, M_{T_t}^n)$ where the random walk process M^n is defined in (4.1), $\mathcal{G}(t, x)$ is a continuous in both variables function and $T(t)$ is a continuous increasing function with $T_0 = 0$. Then for every $\tau \in (0, \infty)$ it holds $(r_t^n)_{0 \le t \le \tau} \to^{\mathcal{D}} (r_t)_{0 \le t \le \tau}$ for $n \to \infty$ where r is defined in (2.2).

The proof of Collorary 4.1 follows from Theorem 4.1 using the fact that $\mathcal{G}(t, [\cdot]_{T_t})$ is a continuous functional in the metric ρ_D on the space $D_{[0,\tau]}(\mathbb{R})$ for every $\tau \in (0, \infty)$.

Remark 4.1. We notice that the functional (1.4) is a special case of the more general functional $\mathcal{G}(t, [\cdot]_{T_t})$ and, hence, the Collorary 4.1 is also true for (1.4).

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