ON ONE-DIMENSIONAL STOCHASTIC EQUATIONS DRIVEN BY SYMMETRIC STABLE PROCESSES

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Abstract

We study stochastic equations $X_t = x_0 + \int_0^t b(u, X_{u-})\, dZ_u$, where Z is an one-dimensional symmetric stable process of index α with $0 < \alpha \leq 2$, $b : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a measurable diffusion coefficient, and $x_0 \in \mathbb{R}$ is the initial value. We give sufficient conditions for the existence of weak solutions. Our main results generalize results of P. A. Zanzotto [18] who dealt with homogeneous diffusion coefficients b. In the nonhomogeneous case we present new sufficient conditons for the existence of (nonexploding) solutions even if Z is a Brownian motion. Using the property that appropriate time changes of stochastic integrals with respect to stable processes are again stable processes with the same index, we present a new proof of the main result which simplifies the approach given by P. A. Zanzotto [18].

Key Words One-dimensional stochastic equations, measurable coefficients, Wiener process, Cauchy process, symmetric stable processes, time change, purely discontinuous processes

MR Subject Classification 60H10, 60J60, 60J65, 60G44

1 Introduction

We consider the stochastic equation

$$X_t = x_0 + \int_0^t b(u, X_{u-})\, dZ_u, \quad t \geq 0, \tag{1.1}$$

where $x_0 \in \mathbb{R}$, $b : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a measurable function, and $(Z, \mathbb{F})$ is an one-dimensional symmetric stable process with $Z_0 = 0$. This is a process with homogeneous and $\mathcal{F}_s$-independent increments $Z_t - Z_s$ for $0 \leq s \leq t$ and right continuous left hand limited (càdlàg) trajectories such that the characteristic function of Z_t has the form

$$\mathbb{E} \exp(i\lambda Z_t) = \exp(-t|\lambda|^\alpha), \quad \lambda \in \mathbb{R}, \quad t \geq 0.$$

Here the parameter α satisfies $0 < \alpha \leq 2$ and is called the index of the symmetric stable process. For $\alpha = 2$, Z is a Brownian motion, the only symmetric stable process with continuous trajectories. For $\alpha = 1$ we obtain the Cauchy process, a càdlàg process Z with homogeneous and independent increments such that Z_t is Cauchy-distributed with parameter t.

It will be useful to have in mind the following equivalent description of symmetric α-stable processes $(Z, \mathbb{F})$: This is a càdlàg process such that, for every $\lambda \in \mathbb{R}$, $\exp(i\lambda Z_t + |\lambda|^\alpha t)$, $t \geq 0$, is a (complex-valued) $\mathbb{F}$-martingale. This can be seen as a substitute for P. Lévy's martingale characterization of Brownian motion.

There are several approaches to the definition of stochastic integrals for symmetric stable processes Z with $0 < \alpha < 2$. Since every symmetric stable process is a semimartingale, the stochastic integral with respect to Z can be understood as a stochastic integral with respect to a semimartingale ([11], Section 4.7). The difficulty is that these processes are purely discontinuous with unbounded jumps and, for $\alpha \leq 1$, even the first moment does not exist. Therefore, in this approach stochastic measures defined by the jumps of the process and their compensators are strongly involved.

Another approach to stochastic integrals with respect to symmetric stable processes is due to J. Rosiński and W. Woyczyński [13] which generalizes the classical concept of K. Itô for Brownian motion to symmetric stable processes.

Of course, both integrals coincide if they exist and we shall make use of the semimartingale integral and the integral introduced by J. Rosiński and W. Woyczyński [13] as well. Whereas the first concept yields general results, in particular, on the time change of such integrals, the advantage of the second concept consists in giving better conditions on the integrands for the existence of the integral.

In this paper we are mainly interested in the existence of weak solutions of Eq. (1.1). The homogeneous Eq. (1.1), i.e., if $b(t,x) = b(x)$, $(t,x) \in [0,\infty) \times \mathbb{R}$, driven by Brownian motion was investigated by the first author and W. Schmidt [5], [6], [7]. There were given necessary and sufficient conditions for existence as well as for uniqueness in law of solutions of Eq. (1.1) for all initial values $x_0 \in \mathbb{R}$. In particular, it was shown that the local integrability of b^{-2} is necessary and sufficient for the existence of nontrivial solutions for all initial values $x_0 \in \mathbb{R}$. Eq. (1.1) with time-dependent coefficients b and $\alpha = 2$ was investigated in several papers ([10], [15], [16], [14]), in which various generalizations of the results mentioned above were obtained. For symmetric stable processes of index $0 < \alpha \leq 2$ Eq. (1.1) with homogeneous coefficients was considered by P. A. Zanzotto [18]. In particular, for the case $1 < \alpha \leq 2$ he proved that there exists a nontrivial solution of Eq. (1.1) for all initial values $x_0 \in \mathbb{R}$ if and only if the function $|b|^{-\alpha}$ is locally integrable. This result is surprising in view of its complete analogy to that obtained in [5] for $\alpha = 2$. For $\alpha \leq 1$ the problem is a little more delicate: For $\alpha < 1$ P.A. Zanzotto gave sufficient conditions, and it seems that necessary conditions can hardly be established. The case $\alpha = 1$ was left open in [18].

The main objective of the present paper is to investigate Eq. (1.1) for symmetric stable processes Z with arbitrary index $\alpha \in (0,2]$ and for general, nonhomogeneous diffusion coefficients b and to give sufficient conditions for the existence of (weak) solutions. We generalize results obtained by P. A. Zanzotto [18] for homogeneous diffusion coefficients b and for indexes α with $0 < \alpha < 2$ and $\alpha \neq 1$. We also include the case if Z is a Cauchy process which corresponds to the case $\alpha = 1$. In particular, we improve the conditions found in [18] in the homogeneous case for indexes α with $0 < \alpha < 1$. Furthermore, even if Z is a Brownian motion, which corresponds to the case $\alpha = 2$, we present new sufficient conditons for the existence of nonexploding solutions for nonhomogeneous diffusion coefficients that are different from those obtained by A. Rozkosz and L. Słomiński [14] and by T. Senf [15], [16]. The basic tool of the paper is time change of symmetric stable processes. Contrary to P. A. Zanzotto [18] we need not examine the jump measures and their compensators associated with symmetric stable processes, their stochastic integrals and time changes. Using the property that appropriate time changes of stochastic integrals with respect to symmetric stable processes are again symmetric stable processes with the same index, we present a new proof of the main result which simplifies the approach given in [18] and unifies the treatment of the continuous case ($\alpha = 2$) on the one side and the purely discontinuous case ($0 < \alpha < 2$) on the other side resulting in a complete analogy between these quite different cases.

In Section 2, we collect some properties on the behaviour of integral functionals of symmetric stable processes which are the key for the construction of the time change processes given in Section 3. These results seem to be of interst in its own right. In Section 4, we briefly review the results of J. Rosiński and W. Woyczyński [13] on stochastic integrals with respect to symmetric stable processes and their time changes giving them a slightly more general form, and we then state sufficient conditions for the existence of (weak) solutions of Eq. (1.1). Finally, in Section 5 we deal with the homogeneous case and, in particular, with the uniqueness in law of solutions.

After submitting the paper, the authors became aquainted with the papers of P.A. Zanzotto [19] and [20], the first being a technical report of the University of Pisa and the second being part of mini-proceedings of the Conference on Lévy processes in Aarhus 1999 appeared in April 1999. It turns out that there is a certain overlap with the present paper concerning the homogeneous case (cf. Section 5). In particular, there is also considered the case of a Cauchy process ($\alpha = 1$). Detailed comments are postponed to Section 5.

2 Integral Functionals of Stable Processes

Let $(S, \mathbb{F})$ be a symmetric stable process with index $\alpha \in (0, 2]$ defined on some probability space $(\Omega, \mathcal{F}, P)$ such that $S_0 = 0$. For any nonegative Borel (or only Lebesgue) measurable function f, we consider the integral functional

$$T_t = \int_0^{t+} f(S_u)\, du, \quad t \geq 0. \tag{2.1}$$

By definition, the process $T = (T_t)_{t \geq 0}$ is increasing and right-continuous taking values in $[0, +\infty]$. First we investigate the case $1 < \alpha \leq 2$. For this we introduce the set

$$E := E(f) = \{x \in \mathbb{R} : \int_{x-}^{x+} f(y)\, dy = +\infty\} \tag{2.2}$$

and note that E is a closed subset of $\mathbb{R}$. We also define the first entry time D_E of S into E by $D_E = \inf\{t \geq 0 : \ S_t \in E\}$. In the sequel, the Lebesgue measure on $\mathbb{R}$ will always be denoted by λ. The following proposition generalizes Lemma 1 of [6] to symmetric stable processes with index $1 < \alpha \leq 2$.

Proposition 2.1 *Suppose that* $1 < \alpha \leq 2$ *and* $\lambda(\{f > 0\}) > 0$. *We then have:*

(i) $T_t < +\infty$ *for all* $t < D_E$ $\quad \mathbf{P}$-*a.s.*

(ii) $T_{D_E} = +\infty \quad \mathbf{P}$-*a.s.*

Proof. Since $1 < \alpha \leq 2$, as for Brownian motion, S has a local time $L^S(t, a)$ jointly continuous in (t, a) and such that $L^S(t, 0) > 0$ **P**-a.s. for every $t > 0$ (cf., e.g., H. Kesten [9], E. S. Boylan [3], Ch. Stone [17]). Moreover, for every $x \in \mathbb{R}$, the first hitting time $\tau_x = \inf\{t > 0 : \ S_t = x\}$ is **P**-a.s. finite (cf. S. C. Port [12]). The proof now follows the same lines as in [6], Lemma 1, and [4], Theorem 1. □

The following 0-1 law is an immediate consequence of Proposition 2.1 and was already stated in [4] for $\alpha = 2$ and by P. A. Zanzotto [18] for $1 < \alpha \leq 2$.

Corollary 2.2 *Suppose that $1 < \alpha \leq 2$. The following conditions are equivalent:*

(i) $\mathbf{P}(\{T_t < +\infty \quad \textit{for every} \quad t \geq 0\}) > 0.$

(ii) $\mathbf{P}(\{T_t < +\infty \quad \textit{for every} \quad t \geq 0\}) = 1.$

(iii) $E = \emptyset$*, i.e., f is locally integrable.*

Corollary 2.3 *Let $1 < \alpha \leq 2$ and $\lambda(\{f > 0\}) > 0$. Then $T_\infty = +\infty$ **P**-a.s.*

Before we investigate the case $\alpha = 1$, let us introduce the following definition.

Definition 2.4 A measurable set $B \subseteq \mathbb{R}$ is called x_0-polar if $\mathbf{P}(\{T(B - x_0) = +\infty\}) = 1$. The set B is called polar if B is x_0-polar for every $x_0 \in \mathbb{R}$.

Here $T(B - x_0)$ is just the first hitting time of $B - x_o$ by S or, equivalently, of B by $x_0 + S$. (In the definition of the first entry time, $\geq$ should be replaced by $>$.) For $0 < \alpha \leq 1$ it is well-known that the singletons $B = \{x\}$, $x \in \mathbb{R}$, and hence all denumerable sets B consisting of isolated points, are polar (cf. [2], Chapter II, (3.15)). In the following, if $0 < \alpha \leq 1$, we frequently use the hypothesis that the set $E = E(f)$ is 0-polar. We emphasize that, in particular, this is satisfied if f is integrable over a sufficiently small neighbourhood of every $x \in \mathbb{R}$, except for, possibly, a denumerable set of isolated points.

Proposition 2.5 *Suppose that $\alpha = 1$ and that the following condition is satisfied:*

$$E \quad \textit{is 0-polar and} \quad \int_{0-}^{0+} |\ln|y|| f(y)\, dy < +\infty. \tag{2.3}$$

Then we have $\quad T_t < +\infty \quad$ *for every* $\quad t \geq 0 \quad$ **P**-*a.s.*

Proof. Let G_N be an increasing sequence of open sets with compact closure $\bar{G}_N \subseteq E^c$ such that $E^c = \bigcup_{N=1}^\infty G_N$ and σ_N defined by

$$\sigma_N = \inf\{t \geq 0 : S_t \in G_N^c\}. \tag{2.4}$$

We note that f is integrable over G_N. Using the quasi-left continuity of S it can easily be seen that σ_N increases to D_E and, since E is 0-polar, to infinity as $N \to \infty$. Hence it is sufficient to verify that, for all $t \geq 0$ and $N \geq 1$, $\mathbb{E}T_{t\wedge\sigma_N} < +\infty$. Using the theorem of Fubini and that S_u is Cauchy-distributed with parameter u we obtain

$$\mathbb{E}\int_0^{t\wedge\sigma_N} f(S_u)\, du \quad \leq \quad \int_0^t \mathbb{E}(1_{G_N}(S_u) f(S_u))\, du$$

$$
\begin{aligned}
&= \int_0^t \int_{G_N} f(y) \frac{u}{\pi(u^2+y^2)}\, dy\, du \\
&= \pi^{-1} \int_{G_N} f(y) \int_0^t \frac{u}{u^2+y^2}\, du\, dy = (2\pi)^{-1} \int_{G_N} f(y) \ln \frac{t^2+y^2}{y^2}\, dy.
\end{aligned}
$$

Now $\ln \dfrac{t^2+y^2}{y^2}$ is continuous at every $y \neq 0$ and behaves like $|\ln|y||$ as $y \to 0$. Hence the right hand side is finite proving the assertion. □

Proposition 2.6 *Suppose that S is a Cauchy process and that $\lambda(\{f > 0\}) > 0$. Then*

$$T_\infty = +\infty \quad \mathbf{P}\text{-}a.s.$$

Proof. For the purpose of this proof, we regard the Cauchy process S as a Markov process defined on a family $(\Omega, \mathcal{F}, \mathbb{F}, P_z, z \in \mathbb{R})$ of filtered probability spaces such that $\mathbf{P}_z(\{S_0 = z\}) = 1$ for every $z \in \mathbb{R}$. For $\varepsilon > 0$ we define

$$\varphi_\varepsilon(z) = \mathbf{P}_z(\{\int_0^\infty f(S_u)\, du > \varepsilon\}), \quad z \in \mathbb{R}.$$

Since $\lambda(\{f > 0\}) > 0$, it can easily be seen that $\mathbb{E}_z \int_0^\infty f(S_u)\, du > 0$ and consequently $\mathbf{P}_z(\{\int_0^\infty f(S_u)\, du > 0\}) > 0$. For some fixed $z \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$, it now follows that $\varphi_\varepsilon(z) > 0$. On the other side, the function φ_ε is excessive and the recurrence properties of S imply that $\varphi_\varepsilon \equiv a$ is constant (cf. R. M. Blumenthal and R. K. Getoor [2], Chapter II, (4.19)). From the Markov property we get

$$
\begin{aligned}
a &= \lim_{N\to\infty} \varphi_\varepsilon(S_N) = \lim_{N\to\infty} \mathbf{P}_{S_N}(\{\int_0^\infty f(S_u)\, du > \varepsilon\}) \\
&= \lim_{N\to\infty} \mathbf{P}_z(\{\int_N^\infty f(S_u)\, du > \varepsilon\} \mid \mathcal{F}_N) = \mathbf{1}_{\bigcap_{N=1}^\infty \{\int_N^\infty f(S_u)\, du > \varepsilon\}} \quad \mathbf{P}_z\text{-a.s.}, \quad z \in \mathbb{R},
\end{aligned}
$$

where the last equality follows from the theorem of Lebesgue-Lévy on the convergence of conditional expectations. But $a > 0$ and, consequently,

$$\mathbf{P}_z(\bigcap_{N=1}^\infty \{\int_N^\infty f(S_u)\, du > \varepsilon\}) = 1, \quad z \in \mathbb{R}.$$

In view of the theorem of Lebesque on majorized convergence

$$\int_0^\infty f(S_u)\, du = +\infty \quad \mathbf{P}_z\text{-a.s.}, \quad z \in \mathbb{R},$$

must hold. Setting $z = 0$ this yields the assertion. □

We notice that the proof also works for indexes $\alpha \in (1, 2]$ giving a second proof of Corollary 2.3. We now consider the case of indexes α with $0 < \alpha < 1$.

Proposition 2.7 *Suppose that $0 < \alpha < 1$ and that the following condition is satisfied:*

$$E \quad \text{is 0-polar and} \quad \int_{0-}^{0+} |y|^{\alpha-1} f(y)\, dy < +\infty. \tag{2.5}$$

We then have $\quad T_t < +\infty \quad$ *for all* $\quad t \geq 0 \quad \mathbf{P}$*-a.s.*

Proof. Let G_N and σ_N be defined by (2.4). As in the proof of Proposition 2.5 it is sufficient to verify that $\mathbb{E}\, T_{t\wedge\sigma_N} < +\infty$. Using (1.7) in Chapter II of [2] we estimate

$$\mathbb{E}\int_0^{t\wedge\sigma_N} f(S_u)\,du \le \mathbb{E}\int_0^{\infty} 1_{G_N}(S_u)\,f(S_u)\,du = c_\alpha \int_{G_N} |y|^{\alpha-1} f(y)\,dy.$$

The right member is finite in view of the definition of G_N and assumption (2.5). □

The next lemma is of interest only for $0 < \alpha \le 1$: For $1 < \alpha \le 2$ it already follows from Proposition 2.1.

Lemma 2.8 *Suppose that $0 < \alpha \le 2$. Let I be a subset of $[0,+\infty)$ containing $[0,t)$ or $[t,+\infty)$ for some $t > 0$. Then*

$$\int_I |S_u|^{-\alpha}\,du = +\infty \quad \mathbf{P}\text{-}a.s.$$

Proof. Let $N \ge 1$ be fixed and set $A_N = \{\int_I |S_u|^{-\alpha}\,du \le N\}$. Using that $u^{-\frac{1}{\alpha}} S_u$ has the same distribution as S_1 we now estimate

$$\begin{aligned} N &\ge \mathbb{E}\left(\mathbf{1}_{A_N}\int_I |S_u|^{-\alpha}\,du\right) = \int_I \mathbb{E}\left(\mathbf{1}_{A_N}(u^{-\frac{1}{\alpha}}|S_u|)^{-\alpha}\right) u^{-1}\,du \\ &= \int_I\int_0^{\infty} \mathbf{P}\left(A_N \cap \{(u^{-\frac{1}{\alpha}}|S_u|)^{-\alpha} \ge v\}\right) dv\, u^{-1}\,du \\ &\ge \int_I\int_0^{\infty} \left(\mathbf{P}(A_N) - \mathbf{P}(\{(u^{-\frac{1}{\alpha}}|S_u|)^{-\alpha} < v\})\right)^+ dv\, u^{-1}\,du \\ &= \left(\int_I u^{-1}\,du\right)\cdot\left(\int_0^{\infty} \left(\mathbf{P}(A_N) - \mathbf{P}(\{|S_1|^{-\alpha} < v\})\right)^+ dv\right). \end{aligned}$$

The first integral on the right hand side is $+\infty$, hence the second integral has to be zero which in view of $\mathbf{P}(\{|S_1|^{-\alpha} > 0\}) = 1$ is only possible if $\mathbf{P}(A_N) = 0$. □

Example 2.9 We have the following properties:

(i) If $\beta \ge \alpha\wedge 1$ then $\int_0^t |S_u|^{-\beta}\,du = +\infty$ for all $t \ge 0$ $\mathbf{P}$-a.s.

(ii) If $\beta < \alpha\wedge 1$ then $\int_0^t |S_u|^{-\beta}\,du < +\infty$ for all $t \ge 0$ $\mathbf{P}$-a.s.

Indeed, for $1 < \alpha \le 2$ and $\beta \ge 1$, (i) follows from Proposition 2.1 and for $0 < \alpha \le 1$ and $\beta \ge \alpha$ from Lemma 2.8 for $I = [0,t]$. Similarly, if $1 < \alpha \le 2$ and $\beta < 1$, (ii) is a consequence of Proposition 2.1 and, if $0 < \alpha \le 1$ and $\beta < \alpha$, of Propositions 2.5 and 2.7. This example shows that $\alpha\wedge 1$ is the critical exponent for convergence or divergence and that, at least for power functions, the conditions (2.3) and (2.5) of Propositions 2.5 and 2.7 cannot be improved.

Finally, we deal with sufficient conditions for $T_\infty = +\infty$ $\mathbf{P}$-a.s. for $0 < \alpha < 1$. Then S is transient in the sense that $\lim_{t\to\infty}|S_t| = +\infty$ $\mathbf{P}$-a.s. Hence the behaviour of f near $-\infty$ and $+\infty$ will be substantial. We introduce the measures μ_α by

$$\mu_\alpha(A) = \int_A |y|^{\alpha-1}\,dy, \quad A \in \mathcal{B}(\mathbb{R}), \quad 0 \le \alpha < 1. \tag{2.6}$$

Lemma 2.10 *Suppose that $0 < \alpha < 1$. We then have:*

(i) *For every measurable subset A of $\mathbb{R}$ such that $\mu_\alpha(A) < +\infty$*

$$\int_0^\infty \mathbf{1}_A(S_u)\,du < +\infty \quad \mathbf{P}\text{-}a.s.$$

In particular, this is true if $\lambda(A) < +\infty$.

(ii) $\lim_{t\to\infty} |S_t| = +\infty \quad \mathbf{P}$-*a.s.*

(iii) $\int_t^\infty |S_u|^{-\alpha}\,du = +\infty$ *for every* $t \geq 0$ $\mathbf{P}$-*a.s.*

(iv) *For every measurable subset A of $\mathbb{R}$ such that $\mu_0(A) < +\infty$*

$$\int_0^\infty \mathbf{1}_A(S_u)\,|S_u|^{-\alpha}\,du < +\infty \quad \mathbf{P}\text{-}a.s.$$

In particular, this is true if $A \subseteq \mathbb{R} \setminus (-1,1)$ and $\lambda(A) < +\infty$.

(v) *For every $\beta > \alpha$*

$$\int_0^\infty \mathbf{1}_{\{|S_u|\geq 1\}})\,|S_u|^{-\beta}\,du < +\infty \quad \mathbf{P}\text{-}a.s.$$

Proof. First we notice that in view of [2] (Chapter II, (1.7))

$$\mathbb{E}\int_0^\infty g(S_u)\,du = c_\alpha \int_{-\infty}^{+\infty} |y|^{\alpha-1}\,g(y)\,dy \tag{2.7}$$

for all nonnegative measurable g. Setting $g = 1_A$, from (2.7) we obtain

$$\mathbb{E}\int_0^\infty 1_A(S_u)\,du = \mu_\alpha(A) < +\infty$$

and hence the occupation time of S in A is finite $\mathbf{P}$-a.s. This verifies (i). For proving (ii), we set $A = (-N, N)$. Part (i) yields $\limsup_{t\to\infty} |S_t| \geq N$ $\mathbf{P}$-a.s. for all $N \geq 1$ and hence $\limsup_{t\to\infty} |S_t| = +\infty$ $\mathbf{P}$-a.s. On the other side, the function φ_x defined by $\varphi_x(y) = |x-y|^{\alpha-1}$, $y \in \mathbb{R}$, is excessive for the symmetric stable semigroup with parameter $0 < \alpha < 1$ (cf. [2], Chapter II, (3.15)). Thus the process $|x - S|^{\alpha-1}$ is a supermartingale if $x \neq 0$ and $\lim_{t\to\infty} |x - S_t|^{\alpha-1}$ exists $\mathbf{P}$-a.s. Consequently, $\lim_{t\to\infty} |S_t| = +\infty$ $\mathbf{P}$-a.s.. This proves (ii). Part (iii) follows from Lemma 2.8 for $I = [t, +\infty)$. For establishing (iv) we use formula (2.7):

$$\mathbb{E}\int_0^\infty \mathbf{1}_A(S_u)\,|S_u|^{-\alpha}\,du = c_\alpha \int_A |y|^{\alpha-1}\,|y|^{-\alpha}\,dy = c_\alpha \int_A |y|^{-1}\,dy = c_\alpha \mu_0(A) < +\infty$$

which, obviously, yields (iv). Analogously, for proving (v) we observe

$$\mathbb{E}\int_0^\infty \mathbf{1}_{\{|S_u|\geq 1\}})\,|S_u|^{-\beta}\,du = 2c_\alpha \int_1^\infty |y|^{\alpha-1}\,|y|^{-\beta}\,dy = 2c_\alpha \int_1^\infty |y|^{\alpha-\beta-1}\,dy < +\infty$$

which completes the proof of the lemma. □

Proposition 2.11 *Suppose that $0 < \alpha < 1$. Let f be a nonnegative measurable function such that the following condition is satisfied:*

$$\exists c > 0: \quad \mu_0(\{x \in \mathbb{R} : |x| \geq 1,\ f(x)\,|x|^\alpha < c\}) < +\infty. \tag{2.8}$$

We then have $T_\infty = +\infty$ $\mathbf{P}$-*a.s.*

Proof. We set $A = \{x \in \mathbb{R} : |x| \geq 1,\ f(x)\,|x|^\alpha < c\}$ and assume $\mu_0(A) < +\infty$ where μ_0 is defined by (2.6). Using Lemma 2.10 (ii) we choose $v = v(\omega)$ so large that $S_u(\omega) \geq 1$ for all $u \geq v$ **P**-a.s. For any $t \geq v$ we now get

$$\begin{aligned} T_t &\geq \int_v^t f(S_u)\,du \geq c\int_v^t \mathbf{1}_{A^c\cap(-1,1)^c}(S_u)\,|S_u|^{-\alpha}\,du \\ &= c\int_v^t \mathbf{1}_{\{|S_u|\geq 1\}}\,|S_u|^{-\alpha}\,du - c\int_v^t \mathbf{1}_A(S_u)\,|S_u|^{-\alpha}\,du \\ &= c\int_v^t |S_u|^{-\alpha}\,du - c\int_v^t \mathbf{1}_A(S_u)\,|S_u|^{-\alpha}\,du, \end{aligned}$$

the right hand side converging to $+\infty$ as $t \to \infty$ **P**-a.s. by Lemma 2.10 (iii), (iv). □

Remark 2.12 Let $0 < \alpha < 1$. Each of the following conditions implies (2.8) and is therefore sufficient for the property $T_\infty = +\infty$ **P**-a.s.:

$$\exists c > 0 : \qquad \lambda(\{x \in \mathbb{R} : f(x)\,|x|^\alpha < c\}) < +\infty. \tag{2.9}$$

$$\exists c > 0 : \qquad \lambda(\{x \in \mathbb{R} : f(x) < c\}) < +\infty. \tag{2.10}$$

$$|x|^{-\alpha} = O(f(x)), \quad |x| \to \infty, \quad \lambda\text{-a.e.} \tag{2.11}$$

The second condition was employed by P. A. Zanzotto (cf. [18], Lemma 2.7).

The following proposition shows, in particular, that the condition (2.8) of Proposition 2.11 (also see (2.9), (2.10), (2.11)) on the asymptotic behaviour of f, in a certain sense, cannot be improved.

Proposition 2.13 *Suppose that $T_t < +\infty$ **P**-a.s. for all $t \geq 0$ (cf. Proposition 2.7 for sufficient conditions). If there exists $\beta > \alpha$ such that the condition*

$$f(x) = O(|x|^{-\beta}), \quad |x| \to \infty, \quad \lambda\text{-}a.e. \tag{2.12}$$

holds then $\quad T_\infty < +\infty \quad$ **P**-*a.s.*

Proof. If (2.12) is satisfied then there exist $N \geq 1$ and $c > 0$ such that

$$f(x) \leq c\,|x|^{-\beta} \quad \text{for all} \quad |x| \geq N \quad \lambda\text{-a.e.}$$

In view of Lemma 2.10 (ii) we can choose $t = t(\omega)$ such that $|S_u(\omega)| \geq N$ for all $u \geq t$. The occupation time in a set of Lebesgue measure zero being zero we obtain

$$\int_0^\infty f(S_u)\,du \leq T_t + c\int_t^\infty \mathbf{1}_{\{|S_u|\geq N\}}|S_u|^{-\beta}\,du.$$

The right hand side is finite **P**-a.s. because of the assumption and Lemma 2.10 (v). □

Remarks 2.14 (i) Suppose that $0 < \alpha < 1$. The symmetry of S and Lemma 2.10 (ii) then imply $\mathbf{P}(\{\lim_{t\to\infty} S_t = +\infty\}) = \mathbf{P}(\{\lim_{t\to\infty} S_t = -\infty\}) = \dfrac{1}{2}$.

(ii) Again let $0 < \alpha < 1$. Then a 0-1 law for the probability $\mathbf{P}(\{T_\infty = \infty\})$ is certainly not satisfied because of (i) above: We can choose $f = 0$ on one half line and bounded below by a strictly positive constant on the remaining half line.

(iii) There are left open several problems concerning the behaviour of integral functionals of symmetric stable processes:

(a) Let S be the Cauchy process. Is then the condition (2.3) of Proposition 2.5 also necessary for $T_t < +\infty$, $t \geq 0$, $\mathbf{P}$-a.s.? Is there valid a 0-1 law as for $1 < \alpha \leq 2$?

(b) In the case $0 < \alpha < 1$, is the condition (2.5) of Proposition 2.7 also necessary for the convergence of the integral functionals T_t for every $t \geq 0$? Because of the transience of S (cf. Lemma 2.10 (iii)) a 0-1 law seems to be not true.

(c) Suppose again $0 < \alpha < 1$. For simplicity we assume that the condition (2.5) is in force ensuring the $\mathbf{P}$-a.s. finiteness of T_t for every $t \geq 0$. A necessary condition for $\mathbf{P}(\{T_\infty = \infty\}) = 1$ is then

$$\int_0^\infty |y|^{\alpha-1} f(y)\, dy = +\infty \quad \text{and} \quad \int_{-\infty}^0 |y|^{\alpha-1} f(y)\, dy = +\infty.$$

Indeed, this easily follows from remark (i) and formula (2.7). Is this condition also sufficient for $\mathbf{P}(\{T_\infty = \infty\}) = 1$?

3 Construction of Time Change Processes

Let $(S, \mathbb{G})$ be a symmetric α stable process defined on $(\Omega, \mathcal{F}, P)$ such that $S_0 = 0$. In this section, we consider the following integral equation:

$$T_t = \int_0^{t+} h(T_u, S_u)\, du, \quad t \geq 0, \tag{3.1}$$

where $h : [0, +\infty) \times \mathbb{R} \to [0, +\infty]$ is a nonnegative measurable function of (t, x). A solution T_t is allowed to take the value $+\infty$ for some $t \geq 0$. For this reason we make the convention $h(+\infty, x) = +\infty$ for every $x \in \mathbb{R}$. We will look for conditions on h guaranteeing that there exist solutions T of Eq. (3.1) which are, moreover, adapted to the filtration $\mathbb{G}^S$ generated by S and completed by all $\mathbf{P}$-null sets. The right inverse A of the increasing process T, defined by

$$A_t = \inf\{s \geq 0 : T_s > t\}, \quad t \geq 0, \tag{3.2}$$

can then serve as a time change for S. This will be carried out in the next section for constructing solutions $(X, \mathbb{F})$ of Eq. (1.1).

For every $N \geq 1$, let the functions $\underline{h}_N$ and $\overline{h}_N$ be defined by

$$\underline{h}_N(x) = \inf_{0 \leq t \leq N} h(t, x), \quad \overline{h}_N(x) = \sup_{0 \leq t \leq N} h(t, x), \quad x \in \mathbb{R}.$$

We suppose that the function h satisfies the following two assumptions:

(A) The function $t \to h(t, x)$ is finite and continuous for λ-a.a. $x \in \mathbb{R}$.

(B) For every $N \geq 1$,

- **(a)** if $1 < \alpha \leq 2$, $\overline{h}_N$ is locally integrable,
- **(b)** if $\alpha = 1$, $f = \overline{h}_N$ satisfies (2.3),
- **(c)** if $0 < \alpha < 1$, $f = \overline{h}_N$ satisfies (2.5).

Theorem 3.1 *Suppose that h satisfies the conditions* **(A)** *and* **(B)**. *Then there exists a $\mathbb{G}^S$-adapted solution T of Eq.* (3.1) *such that $A_t < A_\infty$ on $\{A_\infty < +\infty\}$ for all $t \geq 0$ $\mathbf{P}$-a.s. where A is defined by* (3.2) *and $A_\infty = \sup_{t\geq t} A_t$.*

Proof. First we notice that, instead of **(A)**, we may assume that h is finite and continuous in t *for all* $x \in \mathbb{R}$. Otherwise we find a measurable set $A \subseteq \mathbb{R}$ such that $\lambda(A) = 0$ and $t \to h(t,x)$ is finite continuous for all $x \in A^c$. The function $\tilde{h}$ defined by $\tilde{h}(t,x) = h(t,x)$ if $x \in A^c$ and 0 otherwise for all $t \geq 0$ satisfies the new hypothesis and if we know that there is a solution T of Eq. (3.1) for $\tilde{h}$ then T is also a solution of Eq. (3.1) for h:

$$|\int_0^t h(T_u, S_u)\, du - \int_0^t \tilde{h}(T_u, S_u)\, du| = \int_0^t h(T_u, S_u)\, \mathbf{1}_A(S_u)\, du = 0 \quad \mathbf{P}\text{-a.s.}$$

since the occupation time of S in A is zero $\mathbf{P}$-a.s.
In the second step, we set $\overline{h}(x) = \sup_{t\geq 0} h(t,x)$, $x \in \mathbb{R}$, and assume that, instead of $\overline{h}_N$, $\overline{h}$ satisfies the properties stated in assumption **(B)**. For arbitrary $n \in \mathbb{N}$ we define approximating functions $h_n : [0,\infty) \times \mathbb{R} \to [0,\infty)$ by

$$h_n(t,x) = \inf_{s\in[m2^{-n},(m+1)2^{-n})} h(s,x), \quad \text{if} \quad t \in [m2^{-n},(m+1)2^{-n}), \quad x \in \mathbb{R}.$$

Clearly, h_n is increasing and converges to h pointwise as $n \to \infty$, and $h_n \leq \overline{h}$. In view of the continuity of h in t for every x, we now obtain that $h_n(\cdot,x)$ converges to $h(\cdot,x)$ as $n \to \infty$ *uniformly* on every compact set of $[0,+\infty)$. Because of $h_n \leq \overline{h}$, Corollary 2.2, Proposition 2.5 and Proposition 2.7 we have

$$\int_0^t h_n(m2^{-n}, S_u)\, du \leq \int_0^t \overline{h}(S_u)\, du < +\infty, \quad t \geq 0, \quad \mathbf{P}\text{-a.s.} \tag{3.3}$$

Inductively over m, we now define a sequence $(\tau_m^n)_{m\geq 0}$ of $\mathbb{G}^S$-stopping times:

$$\tau_0^n = 0, \quad \tau_{m+1}^n = \inf\{t \geq \tau_m^n : \int_{\tau_m^n}^t h_n(m2^{-n}, S_u)\, du \geq 2^{-n}\}.$$

Also we set

$$\rho_m^n = \inf\{t \geq 0 : \int_0^t \overline{h}(S_u)\, du \geq m2^{-n}\}, \quad m \geq 0.$$

From (3.3), by induction we observe that $\rho_m^n \leq \tau_m^n$. But $\lim_{m\to\infty} \rho_m^n = +\infty$ $\mathbf{P}$-a.s. which implies $\lim_{m\to\infty} \tau_m^n = +\infty$ $\mathbf{P}$-a.s. for all $n \geq 1$. We now define

$$T_t^n = m2^{-n} + \int_{\tau_m^n}^t h_n(m2^{-n}, S_u)\, du \quad \text{if} \quad \tau_m^n \leq t < \tau_{m+1}^n, \quad m \geq 0.$$

Obviously, T^n is $\mathbb{G}^S$-adapted. It is now elementary to verify that T^n satisfies

$$T_t^n = \int_0^t h_n(T_u^n, S_u)\, du, \quad t \geq 0,$$

and, moreover, $T_t^n \leq T_t^{n+1}$ for all $t \geq 0$. Consequently, there exists the limit T,

$$T_t = \lim_{n\to\infty} T_t^n \leq \int_0^t \overline{h}(S_u)\, du, \quad t \geq 0,$$

and T is $\mathbb{G}^S$-adapted. We show that T is a solution to Eq. (3.1) for the original function h. Since $h_n(\cdot, x)$ uniformly converges to $h(\cdot, x)$ on every compact interval and T_u^n is bounded uniformly in $u \le t$ and n **P**-a.s., we observe

$$\lim_{n\to\infty} h_n(T_u^n, S_u) = h(T_u, S_u), \quad u \le t, \quad \mathbf{P}\text{-a.s.}$$

Moreover, $h_n(T_u^n, S_u)$ is majorized by $\overline{h}(S_u)$ which is integrable over $[0,t]$ **P**-a.s. Using the theorem of Lebesgue on majorized convergence we obtain

$$T_t = \lim_{n\to\infty} T_t^n = \lim_{n\to\infty} \int_0^t h_n(T_u^n, S_u)\, du = \int_0^t h(T_u, S_u)\, du, \quad t \ge 0, \quad \mathbf{P}\text{-a.s.}$$

In the third step, we get rid from the additional assumption of the second step by a localization procedure. For this we define

$$h_N(t,x) = h(t \wedge N, x), \quad (t,x) \in [0,+\infty) \times \mathbb{R},$$

for every $N \ge 1$. The function h_N then satisfies the assumptions of the second step and we consider the $\mathbb{G}^S$-adapted and **P**-a.s. finite solution T^N of Eq. (3.1) for h_N constructed there. Now we set $A_N = \inf\{t \ge 0 : T_t^N > N\}$. By the construction of T^N it is clear that $T_t^N = T_t^{N+1}$ for every $t \le A_N$ **P**-a.s. From this also follows $A_N < A_{N+1}$ on $\{A_N < +\infty\}$ **P**-a.s. Hence the limit $A_\infty = \lim_{N\to\infty} A_N$ exists and $A_N < A_\infty$ on $\{A_\infty < +\infty\}$ **P**-a.s. Finally, for every $t \ge 0$, we put $T_t = T_t^N$ if $t \le A_N$ for some $N \ge 1$ and $+\infty$ if $t \ge A_\infty$. Then T is the required solution of Eq. (3.1). □

In general, the solution T of Eq. (3.1) constructed in the proof of Theorem 3.1 can be bounded or even equal to zero. Roughly speaking, the associated time change A then "explodes" in finite time with positive probability. The following theorem gives conditions that explosions does not occur.

Theorem 3.2 *Let h be a nonnegative measurable function defined on $[0,+\infty) \times \mathbb{R}$. Suppose that h satisfies the following condition:*

(C) *For every $N \ge 1$,*

- **(a)** *if $1 \le \alpha \le 2$, then $\lambda(\{\underline{h}_N > 0\}) > 0$.*
- **(b)** *if $0 < \alpha < 1$, then $f = \underline{h}_N$ satisfies (2.8).*

Let T be any supersolution of Eq. (3.1) for h, i.e., $T_t \ge \int_0^t h(T_u, S_u)\, du$, $t \ge 0$. Then $T_\infty = +\infty$ **P**-*a.s.*

Proof. For every $N \ge 1$, we define $C_N = \{T_\infty \le N\}$. On C_N we obtain

$$N \ge T_t \ge \int_0^t h(T_u, S_u)\, du \ge \int_0^t \underline{h}_N(S_u)\, du.$$

But in view of Corollary 2.3, Proposition 2.6 and Proposition 2.11 the right hand side converges to $+\infty$ **P**-a.s. This yields $\mathbf{P}(C_N) = 0$. □

4 Existence of Solutions of SDEs

For the proof of the main theorem of this section, we first adapt to our needs some basic properties of stochastic integrals with respect to symmetric stable processes due to J. Rosiński and W. Woyczyński [13]. To begin with, we introduce the notion of a symmetric α-stable process stopped at a random time T.

Definition 4.1 Let $(Z,\mathbb{F})$ be a càdlàg process and T an $\mathbb{F}$-stopping time. Then $(Z,\mathbb{F})$ is said to be a symmetric α-stable process stopped at T if for every $\lambda \in \mathbb{R}$ the process $\exp(i\lambda Z_t + |\lambda|^\alpha(t\wedge T))$, $t\geq 0$, is a (complex-valued) $\mathbb{F}$-martingale.

Lemma 4.2 *Every symmetric α-stable process $(Z,\mathbb{G})$ stopped at T can be extended to a symmetric α-stable process $(\tilde{Z},\tilde{\mathbb{G}})$ on $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbf{P}})$ such that $Z_t = \tilde{Z}_{t\wedge T}$, $t\geq 0$.*

Proof. Let $(\bar{Z},\bar{\mathbb{F}})$ be a symmetric α-stable process defined on $(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbf{P}})$ and set $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbf{P}}) = (\Omega,\mathcal{F},\mathbf{P})\times(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbf{P}})$, $\tilde{\mathbb{F}} = \mathbb{F}\times\bar{\mathbb{F}}$ and

$$\tilde{Z}_t(\omega,\bar{\omega}) = Z_{t\wedge T(\omega)}(\omega) + (\bar{Z}_t(\bar{\omega}) - \bar{Z}_{t\wedge T(\omega)}(\bar{\omega}), \quad \tilde{\omega} = (\omega,\bar{\omega})\in\tilde{\Omega}, \quad t\geq 0.$$

One easily checks that then $(\tilde{Z},\tilde{\mathbb{F}})$ is a symmetric α-stable process. □

Proposition 4.3 *Let $(S,\mathbb{G})$ be a symmetric α-stable process. Suppose that H is a $\mathbb{G}$-previsible process with values in $[-\infty,+\infty]$ and set*

$$T_t = \int_0^{t+} |H_u|^\alpha\, du, \quad t\geq 0.$$

Let $A = (A_t)_{t\geq 0}$ be the right inverse of $T = (T_t)_{t\geq 0}$ (see (3.2)). We then have:

(i) *The stochastic integral $\displaystyle\int_0^{A_t} H_u\, dS_u$ is well-defined for all $t\geq 0$.*

(ii) *The process $(Z,\mathbb{F})$ defined by $Z_t = \displaystyle\int_0^{A_t} H_u\, dS_u$, $\mathcal{F}_t = \mathcal{G}_{A_t}$, $t\geq 0$, is a symmetric α-stable process stopped at $U_\infty := T_{A_\infty -}$.*

(iii) *The stochastic integral $\displaystyle\int_0^{t} H_u\, dS_u$ is well-defined on $\{t < A_\infty\}$ for all $t\geq 0$.*

(iv) *If $\mathbf{E}\displaystyle\int_0^t |H_u|^\alpha\, du = 0$ then $\displaystyle\int_0^t H_u\, dS_u = 0$ $\mathbf{P}$-a.s.*

Proof. We only give a brief sketch referring to [13] for more information on stochastic integrals for symmetric α-stable processes. For the previsible process $\mathbf{1}_{[0,A_t]}H$ we have

$$\int_0^\infty |\mathbf{1}_{[0,A_t]}(u)\, H_u|^\alpha\, du = \int_0^{A_t} |H_u|^\alpha\, du = T_{A_t -} = t\wedge T_{A_\infty -} = t\wedge U_\infty \leq t.$$

In the same way as in [13] it can be seen that

$$\int_0^{A_t} H_u\, dS_u := \int_0^\infty \mathbf{1}_{[0,A_t]}(u)\, H_u\, dS_u$$

exists which is realized by using the estimate

$$\sup_{\lambda>0} \lambda^\alpha \mathbf{P}(\{\sup_{0\le t<\infty} |\int_0^t F_u\, dS_u| > \lambda\}) \le c_\alpha \mathbf{E} \int_0^\infty |F_u|^\alpha\, du =: c_\alpha ||F||_\alpha^\alpha \tag{4.1}$$

for every previsible step process F. The inequality (4.1) then extends to every previsible F such that the right hand side is finite. From this, assertion (iv) follows immediately. For proving (ii), we notice that

$$\exp(i\lambda \int_0^t F_u\, dS_u + |\lambda|^\alpha \int_0^t |F_u|^\alpha\, du), \quad t\ge 0,$$

is a bounded $\mathbb{G}$-martingale if F is previsible and such that $\int_0^\infty |F_u|^\alpha\, du$ is bounded. For previsible step processes F this is an easy consequence of the defining property of symmetric α-stable processes, for general F the result follows approximating F by previsible step processes in the norm $||\cdot||_\alpha$. From this follows that

$$\exp(i\lambda \int_0^{t\wedge A_N} H_u\, dS_u + |\lambda|^\alpha \int_0^{t\wedge A_N} |H_u|^\alpha\, du), \quad t\ge 0,$$

is a *bounded* $\mathbb{G}$-martingale and, by Doob's optional sampling theorem,

$$\begin{aligned} &\exp(i\lambda \int_0^{A_t\wedge A_N} H_u\, dS_u + |\lambda|^\alpha \int_0^{A_t\wedge A_N} |H_u|^\alpha\, du) \\ =\ &\exp(i\lambda \int_0^{A_{t\wedge N}} H_u\, dS_u + |\lambda|^\alpha (t\wedge N\wedge U_\infty)), \quad t\ge 0, \end{aligned}$$

is a bounded $\mathbb{F}$-martingale and hence (ii) is proven. Finally, for showing (iii), we set

$$\int_0^t H_u\, dS_u = \int_0^{t\wedge A_N} H_u\, dS_u \quad \text{if} \quad t\le A_N$$

for all $t < A_\infty = \lim_{N\to\infty} A_N$ which correctly defines the desired integral. □

We now come back to Eq. (1.1) and ask for sufficient conditions for the existence of solutions $(X,\mathbb{F})$. Let b be a measurable function defined on $[0,+\infty)\times\mathbb{R}$. If $(X,\mathbb{F})$ is an, in general, exploding solution of Eq. (1.1) we set

$$\tilde{A}_t = \int_0^{t+} |b|^\alpha(u,X_u)\, du \quad \left(= \int_0^{t+} |b|^\alpha(u,X_{u-})\, du\right), \quad t\ge 0, \tag{4.2}$$

and define the right inverse $\tilde{T}$ of the increasing process $\tilde{A}$:

$$\tilde{T}_t = \inf\{s\ge 0 : \tilde{A}_s > t\}, \quad t\ge 0. \tag{4.3}$$

We notice that $\tilde{T}_\infty$ is the explosion time of the increasing process $\tilde{A}$ or, equivalently, of X. This implies $\tilde{A}_{\tilde{T}_\infty -} = +\infty$ on $\{\tilde{T}_\infty < +\infty\}$ **P**-a.s. or, equivalently,

$$\tilde{T}_t < \tilde{T}_\infty \quad \text{on} \quad \{\tilde{T}_\infty < +\infty\} \quad \mathbf{P}\text{-a.s.} \tag{4.4}$$

Otherwise we would have no real explosion but something like a local solution breaking down with a finite left hand limit at $\tilde{T}_\infty$. We also introduce

$$\tilde{U}_\infty = \inf\{s\ge 0 :\ \tilde{A}_s = \tilde{A}_\infty\}. \tag{4.5}$$

Of course, $\tilde{U}_\infty = \tilde{T}_{\tilde{A}_\infty -} \le \tilde{T}_\infty$ and $\tilde{U}_\infty$ is the smallest stopping time such that $\tilde{A}_t = \tilde{A}_{t\wedge \tilde{U}_\infty}$ or, equivalently, $X_t = X_{t\wedge \tilde{U}_\infty}$. Using Proposition 4.3 (i), (ii) (by changing the roles of A and T) we observe that the time changed process $(S,\mathbb{G})$ with

$$S_t = X_{\tilde{T}_t} - x_0 = \int_0^{\tilde{T}_t} b(u, X_{u-})\, dZ_u, \quad \mathcal{G}_t = \mathcal{F}_{\tilde{T}_t}, \quad t \ge 0, \tag{4.6}$$

is a symmetric α-stable process stopped at $\tilde{A}_{\tilde{T}_\infty -} = \tilde{A}_\infty$. Enlarging the probability space, without loss of generality we can, and we always do, assume that $(S,\mathbb{G})$ is extended to a full symmetric α-stable process. Because X is constant $\mathbf{P}$-a.s. on $(t, \tilde{T}_{\tilde{A}_t}]$ which is an interval of constancy for $\tilde{A}$ (cf. (4.1) we obtain the important representation

$$X_t = x_0 + S_{\tilde{A}_t}, \quad t < \tilde{T}_\infty, \quad \mathbf{P}\text{-a.s.} \tag{4.7}$$

Definition 4.4 (i) A solution $(X,\mathbb{F})$ is called *basic* if

$$\int_0^{\tilde{U}_\infty} \mathbf{1}_{\{b=0\}}(u, X_u)\, du = 0 \quad \mathbf{P}\text{-a.s.}$$

(ii) $(X,\mathbb{F})$ is said to be *nonabsorbing* if $\tilde{U}_\infty = \tilde{T}_\infty$ $\mathbf{P}$-a.s.

We notice that $(X,\mathbb{F})$ is nonabsorbing if and only if $\tilde{A}_t < \tilde{A}_\infty$ on $\{\tilde{A}_\infty < +\infty\}$ $\mathbf{P}$-a.s. for every $t \ge 0$.

Lemma 4.5 *Let $(X,\mathbb{F})$ be a solution of Eq.* (1.1)*. We then have:*

(i) *For every $t \ge 0$,*

$$\tilde{T}_t \ge \int_0^t |b|^{-\alpha}(\tilde{T}_u, x_0 + S_u)\, du. \tag{4.8}$$

(ii) *If $(X,\mathbb{F})$ is a basic solution of Eq.* (1.1) *then*

$$\tilde{T}_t \wedge \tilde{U}_\infty = \int_0^{t\wedge \tilde{A}_\infty} |b|^{-\alpha}(\tilde{T}_u, x_0 + S_u)\, du \quad \text{for every} \quad t \ge 0 \quad \mathbf{P}\text{-a.s.} \tag{4.9}$$

and if, additionally, $(X,\mathbb{F})$ is nonabsorbing then

$$\tilde{T}_t = \int_0^t |b|^{-\alpha}(\tilde{T}_u, x_0 + S_u)\, du \quad \text{for every} \quad t \ge 0 \quad \mathbf{P}\text{-a.s.} \tag{4.10}$$

Proof. The continuity of $\tilde{A}$ on $[0, \tilde{T}_\infty)$ and (4.4) yield $\tilde{A}_{\tilde{T}_t} = t \wedge \tilde{A}_\infty$ and, by time change in the integral (see, e.g., [5], Lemma 1.6), we obtain

$$\begin{aligned}\int_0^{t\wedge \tilde{A}_\infty} |b|^{-\alpha}(\tilde{T}_u, x_0 + S_u)\, du &= \int_0^{\tilde{A}_{\tilde{T}_t}} |b|^{-\alpha}(\tilde{T}_u, x_0 + S_u)\, du = \int_0^{\tilde{T}_t} |b|^{-\alpha}(u, X_u)\, d\tilde{A}_u \\ &= \int_0^{\tilde{T}_t} |b|^{-\alpha}(u, X_u)\, |b|^{\alpha}(u, X_u)\, du \le \tilde{T}_t.\end{aligned}$$

Because $\tilde{T}_t = +\infty$ for $t \geq \tilde{A}_\infty$, this proves (4.8). If $(X, \mathbb{F})$ is a basic solution then the inequality becomes an equality $\mathbf{P}$-a.s. on $\{\tilde{T}_t < \tilde{U}_\infty\} = \{t < \tilde{A}_\infty\}$ which yields (4.9) on this set. Passing to the limit $t \uparrow \tilde{A}_\infty$ we obtain

$$\tilde{U}_\infty = \tilde{T}_{\tilde{A}_\infty -} = \int_0^{\tilde{A}_\infty} |b|^{-\alpha}(\tilde{T}_u, x_0 + S_u)\, du \tag{4.11}$$

which is (4.9) on $\{\tilde{T}_t \geq \tilde{U}_\infty\} = \{t \geq \tilde{A}_\infty\}$. If, additionally, $(X, \mathbb{F})$ is nonabsorbing then $\tilde{U}_\infty = \tilde{T}_\infty$ and, in particular, $\tilde{U}_\infty = +\infty$ on $\{\tilde{A}_\infty < +\infty\}$. From this and from (4.9) and (4.11) easily follows that (4.10) is satisfied. □

For every $N \geq 1$, we introduce the functions $\underline{b}_N$ and $\overline{b}_N$ by

$$\underline{b}_N(x) = \inf_{0 \leq t \leq N} |b(t,x)|, \quad \overline{b}_N(x) = \sup_{0 \leq t \leq N} |b(t,x)|, \quad x \in \mathbb{R}.$$

Let $x_0 \in \mathbb{R}$. We shall need the following condition $(\mathbf{E}(x_0))$:

$$(\mathbf{E}(x_0)) \begin{cases} (E_1) & \text{For } \lambda\text{-a.e. } x \in \mathbb{R},\ b(t,x) \text{ is continuous in } t. \\ (E_2) & \text{For every } N \geq 1, \\ & \textbf{(a)} \quad \text{if } 1 < \alpha \leq 2, \text{ then } \underline{b}_N^{-\alpha} \text{ is locally integrable,} \\ & \textbf{(b)} \quad \text{if } \alpha = 1, \text{ then } f = \underline{b}_N^{-\alpha}(x_0 + \cdot) \text{ satisfies (2.3),} \\ & \textbf{(c)} \quad \text{if } 0 < \alpha < 1, \text{ then } f = \underline{b}_N^{-\alpha}(x_0 + \cdot) \text{ satisfies (2.5).} \end{cases}$$

Theorem 4.6 *Suppose that the condition* $(\mathbf{E}(x_0))$ *is satisfied. Then there exists a (possibly, exploding) nonabsorbing basic solution* $(X, \mathbb{F})$ *of Eq.* (1.1) *with* $X_0 = x_0$.

Proof. Let $(S, \mathbb{G})$ be a symmetric α-stable process defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The condition $(\mathbf{E}(x_0))$ ensures that the conditions of Theorem 3.1 for $h = |b|^{-\alpha}(\cdot, x_0 + \cdot)$ are satisfied. (Of course, $|b|^{-\alpha}(t,x) = +\infty$ if $b(t,x) = 0$ for $(t,x) \in [0, +\infty) \times \mathbb{R}$.) For this we notice that $(\mathbf{E}(x_0))$ implies $\lambda(N_b) = 0$ where $N_b = \{x \in \mathbb{R} : \exists t \geq 0 \text{ such that } b(t,x) = 0\}$. (Note that an x_0-polar set is of Lebesque-measure zero.) By Theorem 3.1, there exists an $\mathbb{F}^S$-adapted solution T of

$$T_t = \int_0^{t+} |b|^{-\alpha}(T_u, x_0 + S_u)\, du \quad \left(= \int_0^{t+} |b|^{-\alpha}(T_u, x_0 + S_{u-})\, du \right), \quad t \geq 0,$$

such that

$$A_t < A_\infty \quad \text{on} \quad \{A_\infty < +\infty\} \quad \mathbf{P}\text{-a.s.} \quad \text{for all} \quad t \geq 0. \tag{4.12}$$

Here A denotes the right inverse of the increasing process T defined by (3.2). Obviously, T is strictly increasing and continuous on $[0, A_\infty)$ $\mathbf{P}$-a.s. From this follows that A is strictly increasing and continuous on $[0, T_\infty)$ $\mathbf{P}$-a.s., too. In particular, from this we get the identities $T_{A_t} = t \wedge T_\infty$ and $A_{T_t} = t \wedge A_\infty$, $t \geq 0$. Furthermore, A_t is an $\mathbb{F}^S$-stopping time for every $t \geq 0$. We define the time changed process $(X, \mathbb{F})$ by

$$X_t = \begin{cases} x_0 + S_{A_t} & \text{if } t < T_\infty, \\ \Delta & \text{otherwise} \end{cases}$$

where Δ denotes some fictive point not belonging to $\mathbb{R}$, and $\mathcal{F}_t = \mathcal{G}_{A_t}$ for every $t \geq 0$. Then $(X, \mathbb{F})$ is a càdlàg process. We will show that $(X, \mathbb{F})$ is a solution to Eq. (1.1). Let $N_b = \{x \in \mathbb{R} : \exists t \geq 0 \quad \text{such that} \quad b(t,x) = 0\}$. The condition $(\mathbf{E}(x_0))$ ensures $\lambda(N_b) = 0$. Consequently,

$x_0 + S_t$ has occupation time zero in N_b **P**-a.s. By time change in the integral (see, e.g., [5], Lemma 1.6), we now obtain

$$\begin{aligned} A_t &= \int_0^{A_t} \mathbf{1}_{N_b^c}(x_0+S_u)\,du = \int_0^{A_t} |b|^\alpha(T_u, x_0+S_u)\,|b|^{-\alpha}(T_u, x_0+S_u)\,du \\ &= \int_0^{A_t} |b|^\alpha(T_u, x_0+S_u)\,dT_u = \int_0^{T_{A_t}} |b|^\alpha(u, x_0+S_{A_u})\,du = \int_0^{t\wedge T_\infty} |b|^\alpha(u, X_u)\,du \end{aligned}$$

and, consequently,

$$A_t = \int_0^{t\wedge T_\infty} |b|^\alpha(u, X_u)\,du \quad \text{for all} \quad t \geq 0 \quad \mathbf{P}\text{-a.s.} \tag{4.13}$$

Furthermore, Proposition 4.3 (i), (ii) implies that the stochastic integral

$$Z_t = \int_0^{A_t} b^{-1}(T_u, x_0+S_{u-})\,dS_u, \quad t \geq 0,$$

is well-defined and $(Z, \mathbb{F})$ is a symmetric α-stable process stopped at $U_\infty = T_\infty$. (For this it does not matter that b^{-1} can take the values $\pm\infty$ on some exceptional set.) On the other side, changing the time in the stochastic integral with respect to a semimartingale (see, e.g., [8], Theorem 10.19) yields

$$\begin{aligned} Z_t &= \int_0^{A_t} b^{-1}(T_u, x_0+S_{u-})\,dS_u = \int_0^t b^{-1}(T_{A_u}, x_0+S_{A_u-})\,dS_{A_u} \\ &= \int_0^t b^{-1}(u, X_{u-})\,dX_u, \qquad t < A_\infty, \quad \mathbf{P}\text{-a.s.} \end{aligned} \tag{4.14}$$

From (4.13)

$$\int_0^t |b|^\alpha(u, X_{u-})\,du = \int_0^t |b|^\alpha(u, X_u)\,du = A_t < +\infty, \quad t < T_\infty,$$

and by Proposition 4.3 (iii) (changing the roles of A and T) there exists the stochastic integral $\int_0^t b(u, X_{u-})\,dZ_u$. Because of (4.14) we conclude

$$\begin{aligned} \int_0^t b(u, X_{u-})\,dZ_u &= \int_0^t b(u, X_{u-}) b^{-1}(u, X_{u-})\,dX_u \\ &= \int_0^t 1_{N_b^c}(X_{u-})\,dX_u = X_t - x_0, \quad t < T_\infty, \quad \mathbf{P}\text{-a.s.} \end{aligned} \tag{4.15}$$

where we have used that $\int_0^t \mathbf{1}_{N_b}(X_{u-})\,dX_u = 0$, $t < T_\infty$, **P**-a.s. Indeed, again using Theorem 10.19 of [8] we have

$$\int_0^t 1_{N_b}(X_{u-})\,dX_u = \int_0^{A_t} 1_{N_b}(x_0+S_{u-})\,dS_u$$

and it is sufficient to verify that

$$\int_0^t \mathbf{1}_{N_b}(x_0+S_{u-})\,dS_u = 0, \quad t \geq 0, \quad \mathbf{P}\text{-a.s.}$$

But this follows from Proposition 4.3 (iv) and $\mathbf{E}\int_0^t \mathbf{1}_{N_b}(x_0+S_{u-})\,du = 0$: We have $\lambda(N_b) = 0$ and hence $x_0 + S_{u-}$ has no occupation time in N_b. This proves the claim. Enlarging the probability space, we can assume that Z is extended to a full symmetric α-stable process (not only stopped at T_∞) and, by (4.15), $(X, \mathbb{F})$ is a solution of Eq. (1.1), possibly, exploding in T_∞. The construction of $(X, \mathbb{F})$ and Lemma 4.5 ensure that the solution is basic. Finally, (4.12) yields that $(X, \mathbb{F})$ is nonabsorbing. □

Theorem 4.7 *Suppose that b satisfies the following condition:*

(D) *For every $N \geq 1$,*

- **(a)** *if $1 \leq \alpha \leq 2$, then $\lambda(\{\bar{b}_N < \infty\}) > 0$,*
- **(b)** *if $0 < \alpha < 1$, then $\exists c_N > 0 : \mu_0(\{x \in \mathbb{R} : |x| \geq 1,\ \bar{b}_N(x) > c_N|x|\}) < +\infty$ where μ_0 is defined by (2.6).*

Then every solution $(X, \mathbb{F})$ to Eq. (1.1) *does not explode.*

Proof. Let $(X, \mathbb{F})$ be a solution of Eq. (1.1). We define $\tilde{A}$ by (4.2) and $\tilde{T}$, the right inverse of the increasing process $\tilde{A}$, by (4.3). By (4.6) we know that $(S, \mathbb{G})$ with

$$S_t = X_{\tilde{T}_t} - x_0 = \int_0^{\tilde{T}_t} b(u, X_{u-})\, dZ_u, \quad \mathcal{G}_t = \mathcal{F}_{\tilde{T}_t}, \quad t \geq 0,$$

is a symmetric α-stable process stopped at $\tilde{A}_{\tilde{T}_\infty -}$. By enlarging the probability space, we can assume that $(S, \mathbb{G})$ is extended to a full symmetric α-stable process. The assumptions on b imply that $h = |b|^{-\alpha}(\cdot, x_0 + \cdot)$ satisfies the condition **(C)** of Theorem 3.2. In view of Lemma 4.3 (i), $\tilde{T}$ is a supersolution of Eq. (3.1) for $h = |b|^{-\alpha}(\cdot, x_0 + \cdot)$ and S and Theorem 3.2 now ensures that $\tilde{T}_\infty = +\infty$ **P**-a.s., hence $\tilde{A}_t < +\infty$ **P**-a.s. for all $t \geq 0$ which means that $(X, \mathbb{F})$ is nonexploding. □

We note that the condition $(\mathbf{E}(x_0))$ implies the condition $(\mathbf{D})(\mathbf{a})$. Combining Theorem 4.6 and Theorem 4.7 we therefore get the following

Theorem 4.8 *Suppose that b satisfies the condition* $(\mathbf{E}(x_0))$. *If $0 < \alpha < 1$, additionally we assume that, for every $N \geq 1$,*

$$\exists c_N > 0 : \ \mu_0(\{x \in \mathbb{R} : |x| \geq 1,\ \bar{b}_N(x) > c_N|x|\}) < +\infty. \tag{4.16}$$

Then there exists a nonexploding and nonabsorbing basic solution $(X, \mathbb{F})$ of Eq. (1.1) *with $X_0 = x_0$.*

We recall that, in case of $0 < \alpha < 1$, for the nonexplosion condition (4.16) each of the following conditions is sufficient:

$$\exists c_N > 0 : \qquad \lambda(\{x \in \mathbb{R} : \bar{b}_N(x) > c_N|x|\}) < +\infty. \tag{4.17}$$

$$\bar{b}_N(x) \ = \ O(|x|), \quad |x| \to \infty, \quad \lambda\text{-a.e.} \tag{4.18}$$

The following result shows that, in a certain sense, these conditions cannot be improved: Roughly speaking, if $|b|$ is growing faster than $|x|$ explosion does occur **P**-a.s.

Theorem 4.9 *Let $0 < \alpha < 1$ and $(X, \mathbb{F})$ be a nonabsorbing basic solution of Eq.* (1.1). *Suppose that $\underline{b} := \inf_{N \geq 0} \underline{b}_N$ is such that $\underline{b}^{-\alpha}(x_0 + \cdot)$ satisfies condition* (2.5) *and there exists $\delta > 0$ with*

$$|x|^{1+\delta} = O(\underline{b}(x)), \quad |x| \to \infty.$$

Then X is exploding **P***-a.s.*

Proof. The solution $(X, \mathbb{F})$ is nonabsorbing, hence $\tilde{U}_\infty = \tilde{T}_\infty$ where $\tilde{U}_\infty$ and $\tilde{T}_\infty$ are defined in (4.5) and (4.3). In view of Lemma 4.5) (ii) we have

$$\tilde{T}_\infty = \int_0^\infty |b|^{-\alpha}(\tilde{T}_u, x_0 + S_u)\, du \le \int_0^\infty \underline{b}^{-\alpha}(x_0 + S_u)\, du \quad \mathbf{P}\text{-a.s.}$$

Now $\underline{b}^{-\alpha}(x_0 + \cdot)$ satisfies the conditions of Proposition 2.13 and thus the right member is finite $\mathbf{P}$-a.s. Since $\tilde{T}_\infty$ is the explosion time of X the proof is completed. □

Remark 4.10 In the case $\alpha = 2$, A. Rozkosz and L. Słomiński [14] and T. Senf [15], [16] proved existence of solutions for only measurable diffusion coefficients b. Having this in mind, one should think that condition (E_1) stating the continuity of b in t for a.a. x is too stringent. However, the proofs in [14]–[16] are essentially based on Krylov estimates (as also existence results of Krylov himself), a tool which is not available (at least until now) for $0 < \alpha < 2$. Nevertheless, the existence condition $(\mathbf{E}(x_0))$ is of interest for $\alpha = 2$, too. (Of course, in this case $(\mathbf{E}(x_0))$ does not depend on x_0.) In comparison with the condition of T. Senf [16] who only assumed that b^2 and b^{-2} are locally integrable functions the condition $(\mathbf{E}(x_0))$ has two advantages: Firstly, it guarantess the existence of a *nonexploding* solution. Secondly, $(\mathbf{E}(x_0))$ does not include the local integrability of b^2 at all. We notice that a slightly weaker version of Theorem 4.8 for $\alpha = 2$ is already due to the dissertation of T. Senf [15].

5 The Homogeneous Case

In this section we assume that the diffusion coefficient does not depend on the time and hence is a Borel (or only Lebesgue) measurable function $b : \mathbb{R} \to \mathbb{R}$. This case was extensively studied by P.A. Zanzotto [18]–[20] not only for symmetric but also for skew strictly stable processes where in the latter case b is assumed to be nonnegative. It should be noted that the approach of the present paper also works for skew strictly stable driving processes and nonnegative b. This is based on the fact that Proposition 4.3 remains valid for nonnegative H in this case. For the sake of simplicity, however, we concentrated on the symmetric case.

In comparison with P.A. Zanzotto [18]–[20] we only consider global solutions (and not local solutions). A local solution is only defined until leaving some (small) interval around the starting point. A global solution (which is defined for all $t \ge 0$) is allowed to explode. An exploding solution is in this sense not local: It is a maximal solution exhausting the whole state space. The phenomenon of explosions for homogeneous coefficients b only occurs for parameters $0 < \alpha < 1$ and it was not studied previously.

First we give a reformulation of the results of the last section.

Theorem 5.1 *Let $0 < \alpha \le 2$. If $1 < \alpha \le 2$, suppose that the function $f = |b|^{-\alpha}$ is locally integrable. If $0 < \alpha \le 1$, we assume that $E(f)$ is x_0-polar and, additionally:*

(a) *If $\alpha = 1$, then $\int_{0-}^{0+} |\ln|y||\, |b|^{-1}(x_0 + y)\, dy < +\infty$.*

(b) *If $0 < \alpha < 1$, then $\int_{0-}^{0+} |y|^{\alpha-1}\, |b|^{-\alpha}(x_0 + y) < +\infty$.*

We then have:

(i) *There exists a nonabsorbing basic solution $(X, \mathbb{F})$ with $X_0 = x_0$.*

(ii) *If $1 \le \alpha \le 2$, the solution is nonexploding.*

(iii) *If* $0 < \alpha < 1$ *and* $\exists c > 0: \ \mu_0(\{x \in \mathbb{R} : |x| \geq 1, \ |b|(x) > c|x|\}) < +\infty$ *where* μ_0 *is defined by (2.6) then the solution is nonexploding.*

(iv) *If* $0 < \alpha < 1$ *and there exists* $\delta > 0$ *such that* $|x|^{1+\delta} = O(|b|(x))$, $|x| \to \infty$, *then the solution is exploding* $\mathbf{P}$*-a.s.*

The notion of a nonabsorbing basic solution (cf. (4.4)) is new. Clearly, every nonabsorbing basic solution is nontrivial but not conversely. (A trivial solution X starting at x_0 satisfies $X_t = x_0$ for all $t \geq 0$ $\mathbf{P}$-a.s.) Under the condition of Theorem 5.1, the existence of a nontrivial (nonexploding) solution was proven by H.J. Engelbert and W. Schmidt [5] for $\alpha = 2$ and by P.A. Zanzotto [18] for $1 < \alpha < 2$. In the case $\alpha = 1$, P.A. Zanzotto [19] proved the existence of a nontrivial (nonexploding) solution under the stronger condition $H(x_0)$ (cf. [18]–[20]). In the case $0 < \alpha < 1$, in [19] he proved the existence of a nontrivial and nonexploding solution under the more stringent conditions $H(x_0)$ and (2.10) for $f = |b|^{-\alpha}$. The latter means that, outside of some set of finite Lebesgue measure, b is bounded. The nonexplosion condition (iii) and the explosion condition (iv) in Theorem 5.1 are new. They seem to be sharp as the results of Section 2 suggest (also see Example 5.6 below). For example, a sufficient condition for nonexplosion is that, outside of some set of finite Lebesgue measure, $b(x) = O(|x|)$ as $|x| \to \infty$ and a sufficient condition for explosion is that b is growing faster than $|x|^{1+\delta}$ as $|x| \to \infty$ for some $\delta > 0$.

We recall that the assumption in Theorem 5.1 that $E(f)$ is x_0-polar for $f = |b|^{-\alpha}$ is satisfied if, except for a, possibly, denumerable set of isolated points, $|b|^{-\alpha}$ is integrable over a sufficiently small neighbourhood of $x \in \mathbb{R}$. Thus, if $0 < \alpha \leq 1$, $|b|^{-\alpha}$ is allowed to have a denumerable set of singularities of arbitrary order consisting of isolated points distinct from x_0. However, the condition $H(x_0)$ employed by P.A. Zanzottto [18]–[20] excludes such singularities. Also, for $0 < \alpha \leq 1$ the condition that $|b|^{-\delta}$ is locally integrabble for some $\delta > 1$ (cf. [18] for $0 < \alpha < 1$, [19], Proposition (4.13), for $0 < \alpha \leq 1$) is much more stringent than the conditions used in Theorem 5.1 for every $x_0 \in \mathbb{R}$.

Now we will investigate the uniqueness in law of the solution $(X, \mathbb{F})$ of Eq. (1.1). The next theorem states the uniqueness in law of the nonabsorbing basic solution and is new. The proof is essentially the same as the uniqueness proof of [6], Theorem 2.

Theorem 5.2 *Suppose that* $b : \mathbb{R} \to \mathbb{R}$ *is measurable and* $x_0 \in \mathbb{R}$. *The nonabsorbing basic solution* $(X, \mathbb{F})$ *of Eq. (1.1) satisfying* $X_0 = x_0$ *is unique in law.*

Proof. Let $\tilde{A}$, $\tilde{T}$ and $\tilde{U}_\infty$ be defined by (4.2), (4.3) and (4.5). By Lemma 4.5 (ii)

$$\tilde{T}_t = \int_0^t |b|^{-\alpha}(x_0 + S_u)\, du \quad \text{for every} \quad t \geq 0 \quad \mathbf{P}\text{-a.s.} \tag{5.1}$$

where $(S,\mathbb{G})$ is the symmetric α-stable process defined by (4.6). From (5.1) we now see that $\tilde{T}$ and hence $\tilde{A}$ is a well-defined $\mathcal{G}^S_\infty$-measurable functional. Hence the distribution of the pair $(x_0 + S, \tilde{A})$ is uniquely determined. Recalling the representation (4.7) for X, we conclude that the distribution of X is unique. □

Next we will describe the situation in the case $1 < \alpha \leq 2$ and present necessary and sufficient conditions for existence and uniqueness. Since in this case explosions do not occur, from now on $(X, \mathbb{F})$ always signifies a nonexploding solution of Eq. (1.1). We use the notation $N := N_b = \{x \in \mathbb{R} : \ b(x) = 0\}$ and $E := E(|b|^{-\alpha})$ (cf. (2.2)).

Definition 5.3 A solution $(X, \mathbb{F})$ is called a fundamental solution if

$$\int_0^\infty \mathbf{1}_{N \cap E^c}(X_u)\, du = 0 \quad \mathbf{P}\text{-a.s.}$$

We notice that $(X, \mathbb{F})$ is a fundamental solution if and only if $(X, \mathbb{F})$ is a basic solution and $\tilde{U}_\infty = D_E(X)$ **P**-a.s. (see (4.5) and Definition 4.4).

In the following theorem, the equivalence of (i) and (iii) was also shown by P.A. Zanzotto in [19], Theorem (3.5). For our statement, we exploit the notion of a fundamental solution. The proof of the implication (i)→(ii) is almost the same as for Theorem 4.6, with only slight modifications. Our proof completely unifies the case $1 < \alpha < 1$ on one side and the case $\alpha = 2$ on the other side and is very close to that given by the first author and W. Schmidt in [6].

Theorem 5.4 *Suppose that* $1 < \alpha \leq 2$ *and let* $b : \mathbb{R} \to \mathbb{R}$ *a measurable function. Then the following conditions are equivalent:*

(i) $E \subseteq N$.

(ii) *For every* $x_0 \in \mathbb{R}$*, there exists a fundamental solution* $(X, \mathbb{F})$ *of Eq. (1.1) with* $X_0 = x_0$.

(iii) *For every* $x_0 \in \mathbb{R}$*, there exists a solution* $(X, \mathbb{F})$ *of Eq.(1.1) with* $X_0 = x_0$.

Proof. For proving (i)⇒(ii), as in the proof of Theorem 4.6, let $(S, \mathbb{G})$ be a symmetric α-stable process and we put

$$T_t = \int_0^{t+} |b|^{-\alpha}(x_0 + S_u)\, du, \quad t \geq 0.$$

Let A be the right inverse of T. Then T is strictly increasing and continuous on $[0, A_\infty)$. The $\mathbb{G}^S$-time change A is finite, strictly increasing on $[0, T_{A_\infty -})$ and continuous **P**-a.s. By Proposition 2.1, $A_\infty = D_{E - x_0}$ **P**-a.s. In particular, we have $T_\infty = +\infty$ **P**-a.s. Precisely as in the proof of Theorem 4.6, we now define $(X, \mathbb{F})$ and the symmetric α-stable process $(Z, \mathbb{F})$ stopped at $U_\infty := T_{A_\infty -}$ by

$$X_t = x_0 + S_{A_t}, \quad t \geq 0,$$

and

$$Z_t = \int_0^t b^{-1}(x_0 + S_{u-})\, dS_u, \quad t \geq 0.$$

(For this it does not matter that b^{-1} can take the values $\pm\infty$ on some exceptional set.) Since $A_\infty = D_{E - x_0}$ and A is finite and continuous we can conclude

$$\begin{aligned} U_\infty &= \inf\{t \geq 0 : A_t = A_\infty\} = \inf\{t \geq 0 : A_t = D_{E - x_0}\} \\ &= \inf\{t \geq 0 : x_0 + S_{A_t} \in E\} = \inf\{t \geq 0 : X_t \in E\} = D_E(X). \end{aligned}$$

Using this equality, now we get

$$A_t = \int_0^t |b|^\alpha(X_u)\, du, \quad t \geq 0.$$

The proof is similar to that of (4.13) (and exactly the same as in[6], Theorem 1) and therefore omitted. By Proposition 4.3 (iii) (changing the roles of A and T), the stochastic integrals

$$\int_0^t b(X_u)\, dZ_u, \quad t < T_\infty = +\infty,$$

exist, and we may conclude

$$\int_0^t b(X_{u-})\,dZ_u = \int_0^t \mathbf{1}_{N^c}(X_{u-})\,dX_u, \quad t \ge 0,$$

or, equivalently,

$$\int_0^{t\wedge U_\infty} b(X_{u-})\,dZ_u = \int_0^{t\wedge U_\infty} \mathbf{1}_{N^c}(X_{u-})\,dX_u, \quad t \ge 0.$$

Enlarging the probability space, without loss of generality we extend $(Z, \mathbb{F})$ to a full symmetric α-stable process again denoted by $(Z, \mathbb{F})$. Since $b(x) = 0$ for $x \in E$, we can write

$$\int_0^t b(X_{u-})\,dZ_u = \int_0^{t\wedge U_\infty} \mathbf{1}_{N^c \cap E^c}(X_{u-})\,dX_u, \quad t \ge 0.$$

From the definition of E, $\lambda(N \cap E^c) = 0$ and, as in the proof of Theorem 4.6,

$$\int_0^t \mathbf{1}_{N\cap E^c}(X_{u-})\,dX_u = 0, \quad t \ge 0, \quad \mathbf{P}\text{-a.s.}$$

This yields

$$\int_0^t b(X_{u-})\,dZ_u = \int_0^{t\wedge U_\infty} \mathbf{1}_{E^c}(X_{u-})\,dX_u = \int_0^{t\wedge U_\infty} dX_u = X_{t\wedge U_\infty} - X_0 = X_t - x_0$$

for all $t \ge 0$. It remains to show that $(X, \mathbb{F})$ is fundamental:

$$\begin{aligned} \int_0^\infty \mathbf{1}_{N\cap E^c}(X_u)\,du &= \int_0^\infty \mathbf{1}_{N\cap E^c}(x_0 + S_u)\,dT_u \\ &= \int_0^\infty |b|^{-\alpha}(x_0 + S_u)\mathbf{1}_{N\cap E^c}(x_0 + S_u)\,du = 0 \quad \mathbf{P}\text{-a.s.} \end{aligned}$$

since $\lambda(N \cap E^c) = 0$ and hence $x_0 + S$ has occupation time zero in $N \cap E^c$. The implication (ii)$\Rightarrow$(iii) is trivial. To verify (iii)$\Rightarrow$(i), let $x_0 \in E$. We consider a solution $(X, \mathbb{F})$ of Eq. (1.1) such that $X_0 = x_0$. Using Lemma 4.5 (i) we get

$$\tilde{T}_t \ge \int_0^{t+} |b|^{-\alpha}(x_0 + S_u)\,du, \quad t \ge 0,$$

where $\tilde{T}$ and S are defined by (4.3) and (4.6). From Proposition 2.1, the right hand side is equal to $+\infty$ $\mathbf{P}$-a.s., hence $\tilde{T}_t = +\infty$ $\mathbf{P}$-a.s. for all $t \ge 0$ which implies that for the right inverse $\tilde{A}$,

$$\tilde{A}_t = \int_0^{t+} |b|^\alpha(X_u)\,du, \quad t \ge 0,$$

(see (4.2)) we have $\tilde{A}_t = 0$ $\mathbf{P}$-a.s. for all $t \ge 0$. This is only possible if $X_t = x_0$ $\mathbf{P}$-a.s. for every $t \ge 0$ and hence $b(x_0) = 0$. This proves $E \subseteq N$. □

Finally, we come to the uniqueness in law for $1 < \alpha \le 2$. In the next theorem, part (ii) is due to H.J. Engelbert and W. Schmidt [4] (Theorem 2) for $\alpha = 2$ and to P.A. Zanzotto [19] (Theorem (3.21)) for $1 < \alpha < 2$. The proof is exactly the same as in the Brownian case (cf. [6]) and will only be given for the sake of self-containedness. For $\alpha = 2$, part (i) can be found in [5] (Theorem (5.4)) in the case $E = \emptyset$ or in [7] (Theorem (4.22)) in a slightly different context.

Theorem 5.5 *Suppose that $1 < \alpha \leq 2$ and let $b : \mathbb{R} \to \mathbb{R}$ be a measurable function. We then have:*

(i) *The fundamental solution $(X, \mathbb{F})$ of Eq. (1.1) with $X_0 = x_0$, if it exists, is unique in law.*

(ii) *Suppose that $E \subseteq N$. Then, for every $x_0 \in \mathbb{R}$, the solution $(X, \mathbb{F})$ with $X_0 = x_0$ is unique in law if and only if $E = N$.*

Proof. For proving (i), let $(X, \mathbb{F})$ be a fundamental solution and $\tilde{T}$ and $(S, \mathbb{G})$ be defined by (4.3) and (4.6). First we show

$$\tilde{T}_t = \int_0^{t+} |b|^{-\alpha}(x_0 + S_u)\, du \quad \text{for every} \quad t \geq 0 \quad \mathbf{P}\text{-a.s.} \tag{5.2}$$

In view of Lemma 4.5 (i) and Proposition 2.1 this relation holds on $\{t \geq D_{E-x_0}\}$. Furthermore, we have $D_E(X) \leq \tilde{U}_\infty$ $\mathbf{P}$-a.s. Indeed, from $\tilde{A}_t = \tilde{A}_{\tilde{U}_\infty \wedge t}$ it follows $b(X_u) = 0$ on $[\tilde{U}_\infty, +\infty)$ $\lambda \times \mathbf{P}$-a.e. and, since $(X, \mathbb{F})$ is a fundamental solution, $X_{\tilde{U}_\infty} \in E$ on $\{\tilde{U}_\infty < +\infty\}$ $\mathbf{P}$-a.s. This proves the assertion. Using this and Lemma 4.5 (i) and (ii), $\mathbf{P}$-a.s. on $\{D_E(X) = +\infty\}$ we observe

$$\tilde{T}_t = \int_0^{t \wedge \tilde{A}_\infty} |b|^{-\alpha}(x_0 + S_u)\, du \leq \int_0^t |b|^{-\alpha}(x_0 + S_u)\, du \leq \tilde{T}_t$$

which proves (5.2) on this set. Finally, since $\tilde{A}$ is continuous, on $\{D_E(X) < \infty\}$

$$\begin{aligned} \tilde{A}_{D_E(X)} &= \inf\{\tilde{A}_t : X_t \in E\} = \inf\{\tilde{A}_t : x_0 + S_{\tilde{A}_t} \in E\} \\ &= \inf\{0 \leq u \leq \tilde{A}_\infty : x_0 + S_u \in E\} = \inf\{u \geq 0 : x_0 + S_u \in E\} = D_{E-x_0} \end{aligned}$$

and hence $D_{E-x_0} \leq \tilde{A}_\infty$ on this set. This implies

$$\{D_E(X) < +\infty, t < D_{E-x_0}\} \subseteq \{t < \tilde{A}_\infty\}.$$

Once again using Lemma 4.5 (ii) and the identity $\{t < \tilde{A}_\infty\} = \{\tilde{T}_t < \tilde{U}_\infty\}$ we conclude that (5.2) is also true on the set $\{D_E(X) < +\infty, t < D_{E-x_0}\}$. This completes the proof of (5.2). Now (5.2) shows that $\tilde{T}$ is a $\mathcal{G}^S_\infty$-measurable functional. Exactly as in the proof of Theorem 5.2, from this the uniqueness in law of $(X, \mathbb{F})$ with given initial value x_0 is established. Finally, we prove (ii). If $E = N$ then every solution is a fundamental solution. Using part (i) we see that, for every $x_0 \in \mathbb{R}$, the solution with initial value x_0 is unique in law. Conversely, suppose $E \subseteq N$ and $x_0 \in E^c \cap N$. Then there is a trivial solution X with $X_t = x_0$ for all $t \geq 0$ and there is, by Theorem 5.4, a fundamental solution X with $X_0 = x_0$. These solutions have different laws and hence the uniqueness fails. □

In conclusion, we give the following instructive example extending Examples (4.23) and (4.24) of [19].

Example 5.6 We consider the power function b with

$$b(x) = |x|^\gamma, \quad x \in \mathbb{R}, \quad x \neq 0, \quad \gamma \in \mathbb{R}.$$

Case $\mathbf{1} < \alpha \leq \mathbf{2}$, $\mathbf{b(0) = 0}$: From Theorem 5.4 we get:

1) *For any $x_0 \in \mathbb{R}$ there exists a fundamental solution X starting at x_0. Stopping X at $D_{\{0\}}(X)$ yields also a solution. In particular, there is a trivial solution starting at 0. For every solution, explosion does not occur.*

The uniqueness statement of Theorem 5.5 can be sharpened.

2) *Let $x_0 \in \mathbb{R}$ be fixed. The solution starting at x_0 is unique if and only if $\frac{1}{\alpha} \le \gamma$. In this case, the trivial solution is the (pathwise) unique solution starting at* 0.

For proving this, first we assume $\frac{1}{\alpha} \le \gamma$. Then $E = N = \{0\}$ and the uniqueness of the solution starting at x_0 follows from Theorem 5.5. Conversely, let $\gamma < \frac{1}{\alpha}$. Now let X be the fundamental solution starting at x_0. As the next property **3)** shows, X reaches 0 $\mathbf{P}$-a.s. Stopping X at $D_{\{0\}}(X)$ we obtain a second solution with a different law. Hence uniqueness in law fails.

3) *If $\gamma < \frac{1}{\alpha}$, then every solution X reaches* 0 $\mathbf{P}$*-a.s.*

Indeed, in view of (4.7) we have the representation

$$X_t = x_0 + S_{\tilde{A}_t} \quad \text{for all} \quad t < \tilde{T}_\infty = +\infty, \tag{5.3}$$

where S is a symmetric α-stable process and $\tilde{A}$ is defined by (4.2). Now, as in the proof of Lemma 4.5, on $\{D_{\{0\}}(X) = +\infty\}$

$$\tilde{T}_t = \int_0^t |b|^{-\alpha}(x_0 + S_u)\, du \quad \text{for every} \quad t \ge 0 \quad \mathbf{P}\text{-a.s.}$$

and, using Corollary 2.2, we observe that $\tilde{T}_t$ is finite for every $t \ge 0$, hence $\tilde{A}_\infty = +\infty$ $\mathbf{P}$-a.s. on this set. Consequently, since $x_0 + S$ reaches 0 in finite time $\mathbf{P}$-a.s. (cf. proof of Proposition 2.1) and $\tilde{A}$ is finite and continuous, (5.3) yields that this is also true for X on $\{D_{\{0\}}(X) = +\infty\}$ which is, however, only possible if $\mathbf{P}(\{D_{\{0\}}(X) = +\infty\}) = 0$.

This result can be made more precise in the following way:

4) *For any solution starting at $x_0 \neq 0$, the point* 0 *will be reached with probability* 1 *(resp.,* 0*) if and only if $\gamma < 1$ (resp., $1 \le \gamma$).*

Actually, 0 will be reached with probability 1 (resp., 0) if and only if (say, $x_0 > 0$)

$$\int_0^{x_0} |x|^{\alpha-1} |b(x)|^{-\alpha}\, dx < +\infty \quad (\text{resp.}, = +\infty)$$

from which **4)** derives. For $\alpha = 2$, see S. Assing and T. Senf [1]. For the general case $1 < \alpha \le 2$, a proof will be published in a forthcoming paper elsewhere.

Case $\mathbf{1} < \alpha \le \mathbf{2}$, $\mathbf{b(0)} \neq \mathbf{0}$: There is an essential difference to the case $b(0) = 0$ treated above: The trivial solution starting at 0 is now excluded. Therefore the following properties are true:

5) *There exists a solution starting at* 0 *if and only if $\gamma < \frac{1}{\alpha}$.*

6) *There exists a solution starting at $x_0 \neq 0$ if and only if $\gamma \notin [\frac{1}{\alpha}, 1)$.*

Indeed, if $\gamma \in [\frac{1}{\alpha}, 1)$, then every solution X starting at $x_0 \neq 0$ would reach 0 $\mathbf{P}$-a.s. but it cannot be continued after $D_{\{0\}}(X)$. Thus there is no solution. On the other side, let X be a fundamental solution for the coefficient b with $b(0) = 0$. If $\gamma > 1$, then X does not reach 0 (cf. **4)**) and hence X is also a solution for the present situation. If, however, $\gamma < \frac{1}{\alpha}$ then X reaches 0 $\mathbf{P}$-a.s. but has no occupation time in 0 $\mathbf{P}$-a.s. Again, X is a solution for the coefficient modified in 0.

7) *Every solution is fundamental and hence unique in law.*

Case $\mathbf{0} < \alpha \leq \mathbf{1}$**,** $\mathbf{b(0) = 0}$:

8) *For any starting point* $x_0 \in \mathbb{R}$*, there exists a solution.*

For $x_0 \neq 0$ this follows from Theorem 5.1, for $x_0 = 0$ there is at least the trivial solution.

9) *The trivial solution starting at* 0 *is the (even pathwise) unique solution if and only if* $1 \leq \gamma$*.*

10) *There is a second, nonabsorbing and basic solution starting at* 0 *if and only if* $\gamma < 1$*. This solution is unique in law.*

Indeed, Lemma 4.5 (i) and Example 2.9 show that the trivial solution is the unique solution starting at 0 if $1 \leq \gamma$. On the other side, from Theorem 5.1 we observe that there is a nonabsorbing and basic solution starting at 0 if $\gamma < 1$. The uniqueness statement in **10)** follows from Theorem 5.2.

Now we consider the behaviour of a solution X starting at $x_0 \neq 0$. Since 0 is x_0-polar for the symmetric α-stable process S $(0 < \alpha \leq 1)$, from (5.3) we see that X does not reach 0 **P**-a.s. But $b(x) \neq 0$ for all $x \neq 0$ and, consequently, X is nonabsorbing and basic. Property **8)** and Theorem 5.2 now show:

11) *There is a unique solution starting at* $x_0 \neq 0$*. This solution is nonabsorbing and basic and it does not reach* 0 **P***-a.s.*

This completely clarifies existence and uniqueness of solutions. What about explosion or non-explosion?

12) *If* $\alpha = 1$*, explosion does not occur* **P***-a.s.*

This follows from Theorem 5.1 together with the above uniqueness conclusions **10)** and **11)**. Now we assume $0 < \alpha < 1$. If $\gamma \leq 1$ then $b(x) = O(|x|)$, $|x| \to \infty$, and by Theorem 5.1 (iii) (together with the above uniqueness conclusions **10)** and **11)**) every solution is nonexploding. Furthermore, if $1 < \gamma$, then $|x|^{1+\delta} = O(|b(x)|$, $|x| \to \infty$, for $0 < \delta \leq 1 - \gamma$ and by Theorem 5.1 (iv) the solution starting at $x_0 \neq 0$ is exploding **P**-a.s. Also, if $1 < \gamma$, the solution starting at 0 is trivial (cf. **9)**). This gives:

13) *If* $0 < \alpha < 1$*, the solution is exploding* **P***-a.s. if and only if* $x_0 \neq 0$ *and* $1 < \gamma$*. Otherwise the solution is nonexploding* **P***-a.s.*

Case $\mathbf{0} < \alpha \leq \mathbf{1}$**,** $\mathbf{b(0) \neq 0}$: The results are similar, with only one essential difference: The trivial solution starting at 0 is now excluded. Consequently, we can state:

14) *There is a solution starting at* 0 *if and only if* $\gamma < 1$*. In this case, every solution is nonexploding and fundamental. Hence the solution is unique.*

15) *There is a unique solution starting at* $x_0 \neq 0$*. This solution does not reach* 0 **P***-a.s. Moreover, the solution is exploding* **P***-a.s. if* $0 < \alpha < 1$ *and* $1 < \gamma$*. Otherwise the solution is nonexploding* **P***-a.s.*

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